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Construction of Exact Invariants for Time Dependent Classical Dynamical Systems

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In the present work, we survey various methods used for the construction of exact invariants for dynamical systems involving an explicit time dependence. More stress is placed on two-dimensional (2D) than one-dimensional (1D) systems. While both harmonic and anharmonic time-dependent (TD) systems are discussed in the 1D case, the construction of invariants is carried out for several interesting central and noncentral systems in 2D. The method of complexification of two space dimensions is described in detail. The TD coupled oscillator problem, which in an alternative form suggests the generalization of Ermakov systems, is analyzed in greater detail. The available methods in the 2D case provide only the first invariant, and that for a few TD systems. These methods as such are still inadequate as far as the construction of the second invariant is concerned. The role and scope of some of the derived invariants in the context of various physical problems are highlighted. The possibility of extension of some of these methods to 3D TD systems is also discussed.

1. INTRODUCTION

1.1. Study of Time-Dependent (TD) Systems

During the last three decades or so there has been considerable revival of interest in the study of dynamical systems involving an explicit time dependence. This is mainly due to the fact that in various branches not only of physics but also of engineering (particularly mechanical and electrical engineering) an account of the time dependence (preferably in an exact manner) has become desirable since it gives rise to a deeper insight into the underlying phenomena. In particular, such time-dependent (TD) phenomena

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besides mechanics, frequently occur in other contexts (Section 8) such as the motion of a charged particle in a particular electromagnetic field, hydrodynamics, astrophysics, quantum optics, electrical circuits involving TD capacitors and inductors, quantum mechanics, etc. For quite some time TD systems in one dimension (1D), particularly the TD harmonic oscillator (HO) system, has been of much interest, but it is only during the last decade that TD systems in two and higher space dimensions have acquired increasing interest.

If a dynamical system involves an explicit time dependence, then the corresponding Hamiltonian is not a constant of motion and one has to look for other invariants of the system. The study of TD systems deals not only with the construction of these other invariants, but also with finding their physical interpretations if possible and applications as well. Regarding the number of these other invariants for a TD system, according to Whittaker,[1] it is the same as the space dimensionality of the system. If all the invariants of a system exist and become available, then the system is said to be *integrable*. Naturally, the availability of the invariants of a system simplifies the solution of the corresponding equations of motion. On top of all this, the study of TD systems is interesting as a mathematical exercise in its own right.

An exhaustive survey of time-independent (TID) classical dynamical systems was made by Hietarinta[2] and others.[3,4] The use of the Painlevé conjecture for the study of some of these systems has recently been emphasized in the review by Lakshmanan and Sahadevan.[5] The discussion of TD systems has, however, not been covered in these works. It is true that some of the basic tools for studying both TID and TD systems are common in principle, but in practice the study of the latter turns out to be somewhat more difficult. As a result, only simpler situations are often dealt with in the latter case as compared to those in the former one. Still, for the purposes of clarifying the theoretical understanding of certain physical phenomena in light of experimental results (in view of ever-advancing technologies), it becomes desirable to have an up-to-date report on whatever is known about this difficult subject of TD systems. In the present work we make an effort to give such a survey.

From the point of view of mathematical abstraction, no doubt a TD, n-dimensional Hamiltonian system can be replaced[6] by an $(n + 1)$-dimensional Hamiltonian system in which the time appears as a new canonical coordinate, but for the practical applications of the theory of TD dynamical systems a separate account of its time variable is inevitable. This is what we wish to pursue in this review.

1.2. Study of Noncentral (NC) TD Systems

In order to start any type of study of noncentral (NC) forces in two or higher dimensions, the study of (i) the corresponding system in 1D and/or

(ii) central forces in the same dimensions, at least in terms of mathematical techniques, appears to be a prerequisite. It is only after the type (ii) studies that some meaningful (in the physical sense) results can be arrived at which can highlight the importance of considering the underlying NC forces in describing physical phenomena. Therefore, an understanding of the corresponding 1D system is necessary before actually proceeding to study the 2D system.

Sometimes a given 2D system appears to be a noncentral one, but in reality it may not be so. In fact, a suitable coordinate transformation can be used to convert such a system into a separable or into a central one. This simplifies the study of the given 2D system within the framework of the methods developed for 1D systems. To ensure that the potential function $V(x_1, x_2)$, remains NC or nonseparable in the x_1 and x_2 coordinates under the coordinate transformation $\xi = ax_1 + bx_2$ and $\eta = cx_1 + dx_2$, one should have[2] $ac + bd \neq 0$. Clearly, if $V(x_1, x_2) = f(r)$, where $r^2 = x_1^2 + x_2^2$, then the system has radial symmetry. In this work, while 1D systems and central systems in 2D will be studied, some NC TD systems will also be investigated.

1.3. Different Types of Invariants

The notion of invariants is very widely used in different disciplines of mathematics. In this work, the term "invariant" or the "constant of motion" will be used with reference to the time evolution of the dynamical system. Even in this case various types of invariants are talked about in the literature.[2,4] This is mainly because the construction of "exact" invariants of a given dynamical system has remained an intractable problem. In such a situation often "approximate" invariants, either limited to a subspace of the given phase space or to a limited time dependence of the system, are designed. We shall return to some of these discussions later.

For the Hamiltonian systems, an invariant is basically a phase space function which, in general, can have any functional form. For the TID systems, though several mathematical forms have been investigated[2] in the literature and accordingly the invariants have been constructed, somehow the polynomial (in momenta or in velocities) form remains a basic one and is found to be more convenient for this purpose. For TD systems, however, mainly polynomial (in momenta) forms have been investigated. Only recently have several authors[7] studied a rational form of the invariant for 1D TD systems. In the present work we shall restrict ourselves to the study of only the polynomial form of the invariants.

As far as the classification of the dynamical invariants is concerned, they are broadly classified according to their mathematical functional form in momenta. At the next stage, for a given polynomial form the invariants are classified according to the degree of this polynomial, which, in fact, defines the "order" of the invariant. Most of the work on TD systems carried out thus far has focused on the construction of the second-order invariants only.

1.4. Some Formal Remarks about Dynamical Invariants

Although the theory of dynamical invariants is available(8–12) in a more rigorous mathematical language, we restrict ourselves here to some important formal results which will be of immediate use in the subsequent sections. For details we refer to these classic works.(8–11)

(i) Once the Hamiltonian $H(x_i, p_i, t)$ of the system is known, the time evolution of the coordinates and momenta is given by the Hamilton equations of motion,

$$(dx_i/dt) = (\partial H/\partial p_i); \qquad (dp_i/dt) = -(\partial H/\partial x_i) \tag{1}$$

where $i = 1, 2$ for a 2D system.

(ii) If $I(x_i, p_i)$ is another function in the given phase space, then its time evolution is given by

$$(dI/dt) = [I, H]_{\mathrm{PB}} \tag{2}$$

where $[A, B]_{\mathrm{PB}}$ is the Poisson bracket defined as

$$[A, B]_{\mathrm{PB}} = (\partial A/\partial x_i)(\partial B/\partial p_i) - (\partial A/\partial p_i)(\partial B/\partial x_i) \tag{3}$$

If I has to be a constant of motion (invariant), then

$$(dI/dt) = [I, H]_{\mathrm{PB}} = 0 \tag{4}$$

This implies that the constancy of I depends on the Hamiltonian H, and in particular H itself is a constant of motion for TID (autonomous) systems, and so are the functions of H. Functional independence of two functions G and K can be tested(2) by considering the 2×2 (or $2D \times 2D$ for the D-dimensional systems) Jacobian $\partial(G, K)/\partial(x_i, p_i)$; if its rank is two, then G and K are functionally independent, otherwise they are said to be functionally dependent. For TD systems (nonautonomous), however, the time evolution of a phase space function $I(x_i, p_i, t)$ in general is given by

$$(dI/dt) = (\partial I/\partial t) + [I, H]_{\mathrm{PB}} \tag{5}$$

and again for the constancy of I one should have

$$(dI/dt) = (\partial I/\partial t) + [I, H]_{\mathrm{PB}} = 0 \tag{6}$$

as before. Clearly, the condition (4) is a special case of (6).

In the Lagrangian formulation the equations of motion follow from

$$(d/dt)\,(\partial L/\partial \dot{x}_i) - (\partial L/\partial x_i) = 0 \tag{7}$$

where $L = L(x_i, \dot{x}_i, t)$ is the Lagrangian of the system. Besides these Hamiltonian and Lagrangian formulations of classical mechanics, Hamiltonian–Jacobi theory also throws light on the dynamical invariants in terms of action and angle variables. However, we avoid these discussions here and refer to the literature.[13]

(iii) A D-dimensional Hamiltonian system is said to be integrable (in the sense of Liouville[1]), or rather completely integrable, if there exists a system of D functionally independent functions I_n with $n = 1, 2, 3, \ldots, D$ in the given phase space such that

$$[I_k, I_m]_{\mathrm{PB}} = 0$$

or

$$[I_k, I_m]_{\mathrm{PB}} = \sum_{i=1}^{\mathrm{D}} [(\partial I_k/\partial x_i)\,(\partial I_m/\partial p_i) - (\partial I_k/\partial p_i)\,(\partial I_m/\partial x_i)] = 0 \tag{8}$$

for all k and m, i.e., all I_n are in involution with each other. In principle, there can be more than D functionally independent invariants, but they cannot be in involution. The maximum number of TID invariants is D, including the Hamiltonian, and if all of them exist and are globally defined and single-valued, then the system is sometimes called superintegrable.

In the present work, while we shall ensure the integrability of the TD 1D systems by constructing one invariant for them, for the TD 2D systems, however, we shall be able to construct only the first invariant.

1.5. Scope, Summary, and Arrangement of the Review

Since the discussion of the TID aspect of classical dynamical systems has, as mentioned before, been reviewed by several authors[2,4] in one form or another, in the present article we restrict ourselves to the study of the TD aspect of these systems in 1D and 2D. While we shall present the extension of some of the methods used for 1D systems to the case of 2D systems, their extension from 2D to higher dimensions, in spite of their appearing to be trivial, will not be carried out as such except for some passing remarks in the end. It is true that recently an account of the explicit time dependence of some dynamical systems has led[14] to several interesting new features at the quantum level (like Berry's phase, quantum chaos, etc.), we avoid such discussions in this review and confine ourselves to classical aspects. On the other hand, the studies pursued in this work will be an asset to the understanding of these new quantum phenomena. In fact, such quantum-level studies remain rather incomplete in the absence of a thorough understanding of the corresponding classical-level studies. In a way, the scope of this review is highly limited, but such a compilation of ideas will be very useful as far as future studies of dynamical systems at the level of both classical and quantum mechanics are concerned.

In the present work, we survey various methods used for the construction of exact invariants for dynamical systems involving an explicit time dependence. For this purpose we first review the studies carried out for 1D systems. In particular, both harmonic and anharmonic TD systems are discussed in this case. With a view to demonstrating the underlying complexities in the use of available methods for these constructs, especially when they are used for the construction of higher order invariants in higher dimensions, relatively more emphasis is given to the survey of 2D systems than that of 1D ones. For a variety of TD 2D systems we have been able to construct only one invariant of second order. One of the important problems analyzed using various methods in this case is that of the TD coupled oscillator, which in an alternative form also suggests the generalization of Ermakov systems in 2D. While the construction of third- and higher order (in momenta) invariants in both 1D and 2D cases is demonstrated using the rationalization method, it is pointed out that the available methods for the 2D case, in their present form, are inadequate for the purpose of providing the second invariant for these systems. In this context, a way out is suggested within the framework of the rationalization method. Finally, the role and the scope of some of these derived dynamical invariants with reference to their physical interpretation and applications in various branches of physics are briefly discussed.

The arrangement of this paper is as follows: In the next section, we make a brief survey of 1D systems with some additional remarks on the concept of exact and adiabatic invariants. A summary of various methods used for the construction of exact invariants for 1D systems is presented in Section 3. The details of some of these methods are given in Section 4 in the context of 2D systems. Further, somewhat general results for the third- and higher order invariants in the form of "potential" equations are derived in Section 5. A generalization of Ermakov systems and a new class of Ermakov-type systems based on the results of Section 4 are presented in Section 6. In particular, the problem of coupled TD anharmonic and anisotropic oscillators in 2D is investigated in this section. In Section 7 we discuss the integrability of TD systems in 2D, of course without actually ensuring the same for these systems. Some possible interpretations and applications of some of the derived dynamical invariants are presented in Section 8. Finally, concluding remarks are made in Section 9.

2. A SURVEY OF ONE-DIMENSIONAL TD SYSTEMS

2.1. Exact and Approximate (Adiabatic) Invariants

The study of adiabatic invariants has received[11–13,15] considerable attention in the literature often in connection with the motion of charged particles in a particular electromagnetic field and also in cosmological problems (cf. Section 8). Until recently, any account of the time dependence in a system was identified with the concept of adiabatic invariants more or less in an inseparable manner at the level of both classical and quantum mechanics. While at the quantum level such a confusion still persists, at the classical level several methods have been developed in recent years which can throw light on the nature of invariants for these systems. In fact it has become possible to construct exact invariants for a number of TD systems. In particular, a TD harmonic oscillator (HO) system has been very widely studied.

For a dynamical system involving slow variation with respect to time (or for other physical systems in which a physical quantity changes slowly from one state to another with respect to an independent variable), the adiabatic invariants are defined in analogy with the adiabatic process in thermodynamics. For instance, if λ is a TD parameter of the system, then by slow variation we mean that $T(d\lambda/dt) << \lambda$, where T is the period during which λ varies only slightly. In other words, the functional dependence of $\lambda(t)$ on t is bounded above by an exponential function. Such a system is not closed and hence the energy of the system is not conserved. For a TDHO in which $\omega(t)$ (angular frequency) varies slowly with t the adiabatic invariant turns[13] out to be $I = E/\omega$. For a detailed survey of adiabatic invariants we refer to the work of Chandrasekhar[11] and Whiteman.[12] In this survey, however, we shall confine our discussions to exact invariants. Sometimes, even for systems admitting exact invariants, the presence of perturbation allows the construction of approximate invariants. This, in fact, helps in finding the solution of the problem, although in an approximate manner. In this case, however, the question of the degree to which a quantity appears to be a constant during the successive orders of the perturbation parameter remains an interesting one.

Kolsrud[16] studied exact quantum dynamical solutions for a class of TDHO systems by introducing a unitary time-displacement operator. Later, Kruskal[17] developed a general asymptotic theory of nearly periodic classical systems and derived the invariant for the TDHO system. In fact, in order to see a connection between the exact and approximate invariants, one considers the system

$$H = (1/2\varepsilon)[p^2 + \omega^2(t)x^2] \tag{9}$$

for which there exists an invariant

$$I = (1/2)[(x/\rho)^2 + (\rho\dot{x} - \varepsilon\dot{\rho}x)^2] \tag{10}$$

where $\rho = \rho(t)$ satisfies an auxiliary equation

$$\varepsilon^2\ddot{\rho} + \omega^2(t)\rho = \rho^{-3} \tag{11}$$

Equations (10) and (11) define a class of invariants because ρ may be any particular solution of (11). Now, if (11) is solved recursively to give ρ as a series in positive powers of ε, then that value of ρ can be substituted into (10) to give I as a series in ε. For a classical system with real ω, that series for I is the usual adiabatic (approximate) invariant whose leading term is proportional to $\varepsilon H/\omega$. Kruskal's theory may be applied in a closed form to a system expressed in terms of x, $\dot{x}$, and $\rho(t)$ without actually demanding the adiabatic result in the limit of small ε. Further details of the Kruskal theory are left to the interested reader. Lewis[18] first obtained an exact invariant for TDHO and studied the same in the context of both classical and quantum mechanics. For the quantum case the derivation of a simple relation between eigenstates of such an invariant and the solution of the Schrodinger equation has been studied by a number of authors.[19] It may be mentioned that the TD phase associated with the eigenstates of the invariant satisfies a simple first-order nonlinear differential equation.

2.2. Study of One-Dimensional Systems

As mentioned before, the Hamiltonian of a system in 1D involving an explicit time dependence is not a constant of motion and one has to look for the other invariant of the system. The most studied case is that of a TDHO described by the Hamiltonian [dropping ε from equation (9)]

$$H = (1/2)[p^2 + \omega^2(t)x^2] \tag{12}$$

with the corresponding equation of motion

$$\ddot{x} + \omega^2(t)x = 0 \tag{13}$$

The system (12) admits the invariant

$$I = (1/2)[k(x/\rho)^2 + (\rho\dot{x} - \dot{\rho}x)^2] \tag{14}$$

with $\rho(t)$ satisfying

$$\ddot{\rho} + \omega^2(t)\rho = k\rho^{-3} \tag{15}$$

It may be mentioned that the t dependence, although shown in the potential term in (12), may arise through any one or both terms. For example, in physical problems a pendulum with time-varying mass and/or length can give rise[20] to such systems. Also, the damping (if present) can involve a t dependence. Note that the appearance of the constant k in (14) and (15) is illusory. While it may be important for physical reasons (cf. Section 8), it can as well be eliminated by a scale transformation $\rho \to \sqrt{k}\,\rho$. In any case, there exists a transformation (see, for example, ref. 21) which converts a TD damped system into a TD undamped one, and subsequently a rescaling of x and t variables leads to the form (12). For example, the Lagrangian corresponding to the equation of motion (damped case),

$$\ddot{x} + f(t)\dot{x} + \omega^2(t)x = 0 \tag{16}$$

can be expressed as[22]

$$L = (1/2)e^{F(t)}\,[\dot{x}^2 - \omega^2(t)x^2] \tag{17}$$

where $dF/dt = f(t)$. If one defines the conjugate momentum $p = \dot{x}e^F$, then the corresponding Hamiltonian becomes $H = (1/2)[p^2e^{-F} + \omega^2(t)e^Fx^2]$ describing a pendulum with time dependence in both mass and frequency. In general, the system

$$\ddot{y} + f(t)\dot{y} + \overline{\omega}^2(t)y = G(t) \tag{18}$$

with arbitrary TD functions $f(t)$, $\overline{\omega}^2(t)$, and $G(t)$ can be cast in the form[23]

$$\ddot{x} + \omega^2(t)x = g(t) \tag{19}$$

by using the well-known transformation

$$x = y\exp[(1/2)\int^t f \quad dt] \tag{20}$$

and with

$$\omega^2(t) = \overline{\omega}^2(t) - \tfrac{1}{2}\dot{f} - \tfrac{1}{4}f^2; \qquad g(t) = G(t)\exp[\tfrac{1}{2}\int^t f\,dt]$$

The problem of the TD anharmonic oscillator with cubic anharmonicity was investigated by Leach[24] and Maharatna *et al.*[24] and an exact invariant was obtained by Leach using the method of the Lie theory of extended groups. The system he considered is

$$\ddot{x} + a(t)\dot{x} + b(t)x + c(t)x^2 + d(t) = 0 \tag{21}$$

In a paper by Leach and Maharaj[25] the first invariant is constructed also for a class of TD anharmonic oscillators of more complex type, namely the system

$$\ddot{x} = a_1(t)x^2 + a_2(t)x^3 + a_3(t)x^4 + a_4(t)x^5 \tag{22}$$

is studied. In fact, some particular cases of this equation arise in the study of a charged plasma in an axially symmetric magnetic field and in a shear-free spherically symmetric gravitational field in general relativity. Another TD anharmonic system which has been of interest[26,27] is the modified Emden equation,

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0 \ (n = \text{positive integer}) \tag{23}$$

which arises in the study of a spherical gas cloud acting under the central attractions of its molecules and subject to the laws of thermodynamics. The first invariant is constructed for this system by Leach[27] using a Lie point symmetry analysis.

The case of nonlinear equations of motion was also considered by Ray and Reid[28] using Noether's theorem and by Kaushal and Korsch[21] using the dynamical algebraic approach, corresponding to the system

$$L = (1/2)[\dot{x}^2 - \omega^2(t)x^2 - \phi(x, t)] \tag{24}$$

Interestingly, the system is found to admit an invariant for the case when $\phi(x, t) = 2F_0G(t)\,x^{-2m}$, where m is an arbitrary constant, in the method of Ray and Reid. On the other hand, in the dynamical algebraic approach ϕ satisfies a PDE one of whose particular solutions is the same as that obtained by Ray and Reid. The case when ϕ is momentum dependent (instead of x dependent) is also investigated in the dynamical algebraic approach and accordingly an invariant is constructed for the form $\phi(p, t) = 2G_0\, n(t)p^{-2m}$. Besides the above cases, several generalizations of $\phi(x, t)$ in terms of the auxiliary variable $\rho(t)$ have also been considered in the literature.[21,28] We shall return to some of them in the following sections.

3. METHODS FOR ONE-DIMENSIONAL TD SYSTEMS

Several methods have been developed in order to obtain the invariant for TD systems in 1D. Sometimes the system (12) or other, related forms have offered a testing ground for deciding the merit of a method used for this purpose. While it may be more appropriate to discuss the details of these methods in the context of 2D systems in the next section, here it is worth giving a brief summary of them for 1D systems. Besides the rationalization method of Whittaker type (Hietarinta[2] terms this method the "direct" method), the other methods which we wish to emphasize below are the Ermakov method, the dynamical algebraic approach. Lutzky's approach using Noether's theorem (related to the Lie symmetries approach), the transformation-group method, and a few others.

3.1. Rationalization Method

Here one makes[29] an ansatz for the nth-order invariant as

$$I_n = b_0 + b_1\dot{x} + (1/2!)b_2\dot{x}^2 + \ldots + (1/n!)b_n\dot{x}^n \tag{25}$$

where the coefficient functions $b_i \equiv b_i(x, t)$. Note that unlike the TID case, here all powers in $\dot{x}$ appear in I_n up to a given n; this in fact complicates the applicability of the method for the TD case, particularly in higher dimensions, as will be clear from the next section. Now, for the Hamiltonian

$$H = (1/2)p^2 + V(x, t) \tag{26}$$

the use of equations (5) and (6) will yield a recursion relation for the b_i as

$$\dot{b}_i + i(\partial\, b_{i-1}/\partial x) - b_{i+1}(\partial V/\partial x) = 0 \tag{27}$$

where $i = 0, 1, 2, \ldots, n$. We postpone the case of third- and higher order invariants to Section 5; the results given here are those for the first- and second-order invariants.

For the first-order invariant, the PDEs to be solved for b_0 *and* b_1 are

$$(\partial b_1/\partial x) = 0; \qquad (\partial b_0/\partial x) = -(\partial b_1/\partial t); \qquad \partial b_0/\partial t = b_1(\partial V/\partial x)$$

which lead to the "potential"equation

$$(\partial V/\partial x) + (\ddot{\rho}_1/\rho_1)x - (\dot{\rho}_2/\rho_1) = 0 \tag{28}$$

with the only solution

$$V(x, t) = -(\ddot{\rho}_1/2\rho_1)\, x^2 + (\dot{\rho}_2/\rho_1)x + \rho_3(t)$$

This system corresponds to a TD, rotating HO expressed by $V(x, t) = \frac{1}{2}\omega^2(t)[x - \alpha(t)]^2$, and admits the invariant $I = \rho_2 + (\rho_1\dot{x} - \dot{\rho}_1 x)$, where ρ_1 and ρ_2 are functions of t satisfying $\ddot{\rho}_1 + \omega^2(t)\rho_1 = 0$ and $\rho_2 + \omega^2(t)\alpha(t)\rho_1 = 0$.

For the second-order invariant, the PDEs to be solved for b_0, b_1, and b_2 are

$$(\partial b_2/\partial x) = 0 \tag{27a}$$

$$2(\partial b_1/\partial x) + (\partial b_2/\partial t) = 0 \tag{27b}$$

$$(\partial b_0/\partial t) + (\partial b_1/\partial t) - b_2(\partial V/\partial x) = 0 \tag{27c}$$

$$(\partial b_0/\partial t) - b_1(\partial V/\partial x) = 0 \tag{27d}$$

and the "potential" equation turns out to be

$$\left(-\tfrac{1}{2}\,\dot{\sigma}_1 x + \sigma_2\right)(\partial^2 V/\partial x^2) - \sigma_1(\partial^2 V/\partial t\cdot\,\partial x) - \tfrac{3}{2}\,\dot{\sigma}_1(\partial V/\partial x) + \left(-\tfrac{1}{2}\,\dddot{\sigma}_1 x + \ddot{\sigma}_2\right) = 0 \tag{29}$$

where σ_1 and σ_2 are arbitrary functions of t and should be fixed by rationalizing (29) for a given V. Equation (29) is a linear, second-order PDE whose solution, in principle, would provide the integrable systems admitting second-order invariants. Using (29), while it is not difficult to recover the invariant (14) for the system (12), the case of a TD arbitrary power potential, namely $V(x,\, t) = \beta(t)x^m$, can also be analyzed.[29]

3.2. Ermakov Method

The study of a system of coupled, nonlinear second-order oscillators possessing at least one invariant has become interesting from the point of view of applications. Ermakov[30] originally suggested a connection between the solutions of such a pair of coupled equations, hereafter, termed as Ermakov systems. In recent years, Ray and Reid in a series of papers[23,28,31,32] have studied these systems in the context of TDHO and with several degrees of generalization. Ray and Reid have evolved a method of constructing the invariant for TD systems in 1D, known as the Ermakov method, and accordingly the invariant so constructed sometimes is known as the Ermakov invariant. This method, although simple, is a heuristic one and sometimes leads to more general systems possessing invariants.

In this method, one eliminates $\omega^2(t)$ from the equation of motion for a TDHO, viz.,

$$\ddot{x} + \omega^2\,(t)x = 0$$

and the auxiliary equation (15). As a result of the first integration of the resultant equation after multiplying the latter by $(\dot{x}\rho - x\dot{\rho})$, one immediately obtains the invariant (14). In this case, however, one has to know the auxiliary equation in advance. Besides accounting for the damping terms in (13) and (15), the most common generalization considered by Ray and Reid is in terms of the equations

$$\ddot{x} + \omega^2(t)x = g(\rho/x)/(x^2\rho) \tag{30a}$$

$$\ddot{\rho} + \omega^2(t)\rho = f(x/\rho)/(x\rho^2) \tag{30b}$$

These equations, as before, lead to the invariant

$$I = (1/2)[\phi(x/\rho) + \theta(\rho/x) + (x\dot{\rho} - \dot{x}\rho)^2] \tag{31}$$

with

$$\phi(x/\rho) = 2\int^{(x/\rho)} f(u)\,du; \qquad \theta(\rho/x) = 2\int^{(\rho/x)} g(u)\,du$$

We shall return to some of these discussions in Sect. 6.

3.3. Dynamical Algebraic Approach

Earlier Korsch[33] for a limited number of TD systems and later Kaushal and Korsch[21] for a variety of TD Hamiltonian systems exploited the closure property of dynamical Lie algebra generated by the phase-space functions Γ. Takayama[34] applied this approach to obtain the invariant for (19). While the details and extension of this method to 2D systems will be discussed in the next section, here we present the central idea.

In this approach one expresses the Hamiltonian of the system as

$$H = \sum_n h_n(t)\Gamma_n(x, p) \tag{32}$$

where the Γ_n are not explicitly TD. Here the dynamical algebra is the Lie algebra of the Γ_n, which is closed with respect to the Poisson bracket,

$$[\Gamma_n, \Gamma_m]_{\mathrm{PB}} = \sum_r C^r_{nm}\Gamma_r \tag{33}$$

where C^r_{nm} are the structure constants of the algebra. If the Γ_n appearing in (32) are not sufficient to close the algebra, then the set of Γ_n must be extended by the inclusion of new Γ_l such that $\Gamma_l = [\Gamma_n, \Gamma_m]_{\mathrm{PB}}$ [with $h_l(t)$ taken to be zero in (32)] until the closure is obtained. It may be mentioned that the algebra contains important structural information for the dynamical behavior [independent of the particular functions $h_n(t)$ appearing in (32)] of the system besides its straightforward extension (see, for example, Mizrahi[19] and Kaushal and Korsch[21]) to the corresponding quantum case. Since the invariant I is also a phase-space function, and thus is a member of the dynamical algebra, it should be expressible as

$$I = \sum_k \lambda_k(t)\Gamma_k \tag{34}$$

and its time development should be in accordance with (5) and (6). Equations (5) and (6). in fact, imply

$$(\partial I/\partial t) = [H, I]_{\mathrm{PB}} \tag{35}$$

which, after using (32) and (34), will lead[33] to the identity

$$\sum_r [\dot{\lambda}_r + \sum_{n,m} C^r_{nm} h_m(t)\lambda_n(t)]\ \Gamma_r \equiv 0 \qquad (r = 1, 2, \ldots)$$

Clearly, this identity provides a system of linear, first-order ODEs, namely

$$\dot{\lambda}_r + \sum_n [\sum_m C^r_{nm} h_m(t)\lambda_n] = 0 \tag{36}$$

from which the unknown λ_k can be determined. Once the λ_k are known, the invariant can be computed from (34). For further details and applications of the method to specific examples we refer to ref. 21.

3.4. Lutzky's Approach Using Noether's Theorem

This method is based on the following formulation of Noether's theorem due to Lutzky[35] and subsequently used by Ray and Reid[28] for TD systems in 1D. In this approach, modified for the TD case, the symmetry transformation is described by the group operator

$$X = \zeta(\rho, t)\frac{\partial}{\partial t} + \eta(x, t)\frac{\partial}{\partial x} \tag{37}$$

If the symmetry transformation defined by (37) leaves the action A,

$$A = \int L(x, \dot{x}, t)\, dt$$

invariant, then the combination of the terms $\zeta(\partial L/\partial t) + \eta(\partial L/\partial x) + (\dot{\eta} - \dot{x}\dot{\zeta})(\partial L/\partial \dot{x}) + \dot{\zeta}L$ is the total time derivative of a function $f(\rho, t)$, i.e.,

$$\zeta(\partial L/\partial t) + \eta(\partial L/\partial x) + (\dot{\eta} - \dot{x}\dot{\zeta})(\partial L/\partial \dot{x}) + \dot{\zeta}L = \dot{f} \tag{38}$$

It follows from this that a constant of motion for the system is

$$I = (\zeta\dot{x} - \eta)(\partial L/\partial \dot{x}) - \zeta L + f \tag{39}$$

In (38), $\dot{\zeta}$, $\dot{\eta}$, and $\dot{f}$ are defined as

$$\dot{\zeta} = (\partial\zeta/\partial t) + \dot{x}(\partial\zeta/\partial x)$$
$$\dot{\eta} = (\partial\eta/\partial t) + \dot{x}((\partial\eta/\partial x)$$
$$\dot{f} = (\partial f/\partial t) + \dot{x}(\partial f/\partial x)$$

This method is successfully applied not only to the TDHO,[35] but also to several of its generalizations. The results derived are the same as obtained by using the dynamical algebraic approach.

3.5. Transformation-Group Method

With a view to obtaining an exact solution of the Schrödinger equation for a TDHO potential in 1D, the transformation-group method was used by Ray.[36] This method, based on the transformation-group techniques introduced by Burgan *et al.*,[37] essentially deals with the transformation of both dependent and independent variables. The unknown coefficient functions of the transformation are set in such a way that the form of the equation of motion remains invariant under the transformation. Interestingly, the energy integral in the new coordinates turns out to be the desired invariant of the system. Here, we demonstrate the method for the system (19).

For the system (19), we use the transformation

$$x' = x/C(t) + A(t); \qquad t' = D(t) \tag{40}$$

where C, A, and D are arbitrary functions of t. Under this transformation, (19) takes the form[36]

$$C\dot{D}^2 \frac{d^2x'}{dt'^2} + (2\dot{C}\dot{D} + C\ddot{D}) \frac{dx'}{dt'} + [\ddot{C} + \omega^2(t)C]x' + [-\ddot{C}A - 2\dot{C}A - \omega^2(t)CA - C\ddot{A} - g] = 0 \tag{41}$$

Demanding that the form (19) remain invariant under (40), the coefficient of (dx'/dt') in (41) must vanish. This yields $\dot{D} = dt'/dt = 1/C^2$ and accordingly converts (41) into the form

$$\frac{d^2x'}{dt'^2} + C^3[\ddot{C} + \omega^2(t)C]x' + C^3[-\ddot{C}\mathrm{A} - 2\dot{C}A - \omega^2(t)CA - C\ddot{A} - g] = 0 \tag{42}$$

In order to identify (42) with the equation (i.e., with the equation of motion for a TID HO)

$$\frac{d^2x'}{dt'^2} + kx' = 0 \tag{43}$$

one has to chose A and C in (42) such that

$$\ddot{C} + \omega^2(t)C = k/C^3 \tag{44a}$$

$$\ddot{A} + (kA/C^4) + 2(\dot{C}\dot{A}/C) + g/C = 0 \tag{44b}$$

The energy integral for (43) has the form

$$I = \tfrac{1}{2}[(dx'/dt')^2 + kx'^2]$$

which, after carrying out the inverse transformation, reduces to the form

$$l = \tfrac{1}{2}(C\dot{x} - \dot{C}x + C^2\dot{A})^2 + \tfrac{1}{2}k(x/c + A)^2 \tag{45}$$

where C is any solution of (44a) and A is any solution of (44b). The invariant (45) and equations (44) are the same as derived by Takayama[34] using the dynamical algebraic approach. Leach[24] also employed the transformation (40) to find the invariants for some autonomous systems.

3.6. Other Methods

Besides the methods mentioned above, several other methods have also been used to construct the invariant for TD systems in 1D. In this regard, the method of self-similar techniques as used by Feix *et al.*[38] is worth mentioning. The underlying idea of this method is rather simple and can be expressed as follows:

For the system (26) one looks for the invariant $I(x, p, t)$ in accordance with (5) and (6), which after using $\dot{x} = (\partial H/\partial p)$ and $\dot{p} = -(\partial H/\partial x)$ reduce to the form

$$(\partial I/\partial t) + \dot{x}(\partial I/\partial x) + \dot{p}(\partial I/\partial p) = 0 \tag{46}$$

Again, after using $\dot{x} = p$ and $\dot{p} = -(\partial V/\partial x) = \mathcal{F}(x, t)$, this equation can be cast in the form

$$\mathcal{F}(x, t) = -[(\partial I/\partial t) + p(\partial I/\partial x)]/(\partial I/\partial p) \tag{47}$$

Further, the differentiation of this equation w.r.t. p (keeping in mind that the left-hand side is independent of p) leads to

$$\begin{aligned}(\partial I/\partial p)\,[(\partial^2 I/\partial t\cdot\partial p) + (\partial I/\partial x) + p(\partial^2 I/\partial x\cdot\partial p)]\\ - (\partial^2 I/\partial p^2)[(\partial I/\partial t) + p(\partial I/\partial x)] = 0\end{aligned} \tag{48}$$

Now one assumes that the solutions of (48) are self-similar, i.e., one chooses to absorb the time in the reduced variables proportional to x and p in such a way that the (so-called) self-similar transformation[38]

$$t = a^{\alpha}\,\bar{t}; \qquad x = a^{\beta}\bar{x}; \qquad p = a^{\gamma}\bar{p}; \qquad I = a^{\delta}\bar{I} \tag{49}$$

leaves (48) unchanged and thus yielding the relation $2\delta - \alpha - 2\gamma = 2\delta - \beta - \gamma$, thereby leaving only three of α, β, γ, and δ as arbitrary. Next, after introducing the parameters λ and μ through $\beta/\alpha = \lambda$ and $\delta/\alpha = \mu$ and defining the new variables in terms of λ and μ as

$$\xi = x/t^{\lambda}; \qquad \eta = p/t^{\lambda-1}; \qquad G = I/t^{\mu} \tag{50}$$

one can immediately write down the expressions for $(\partial I/\partial t)$, $(\partial I/\partial x)$, and $(\partial I/\partial p)$ in terms of these new variables. The use of these expressions in (47) gives

$$\begin{aligned}\mathcal{F}(x, t) &= -t^{\lambda-2}[\mu G + (\eta - \lambda\xi)(\partial G/\partial\xi) + (1 - \lambda)\eta(\partial G/\partial\eta)]/(\partial G/\partial\eta)\\ &= t^{\lambda-2}\, F(\xi)\end{aligned} \tag{51}$$

where $F(\xi)$ is the "reduced" force. One can define the "reduced" potential $\phi(\xi)$ through $F(\xi) = -(d\phi/d\xi)$ and express the potential term in (26) as

$$V(x, t) = t^{2\lambda-2}\,\phi(x/t^{\lambda}) \tag{52}$$

Finally, equation (51) can be cast in the form

$$\mu G + \omega(\partial G/\partial\xi) + [\tilde{F} + \omega(1 - 2\lambda)](\partial G/\partial\omega) = 0 \tag{53}$$

with $\omega = \eta - \lambda\xi$ and $\tilde{F} = F + \lambda(1 - \lambda)\xi$.

Since G is directly related to the invariant I [see equation (50)], the solution of (53) will immediately provide the invariant for the system (26). Not only this, different ansatze for the solution to equation (53), namely $G = \omega + a(\xi)$, $G = \omega^2 + a(\xi)\omega + b(\xi)$, $G = \omega^2 + a(\xi)\omega^2 + b(\xi)\omega + c(\xi), \ldots,$ as considered by Feix *et al.*[38] will give rise to invariants of different orders in momenta. Interestingly, the class of potentials $V(x, t) = U(x/C)/C^2 - (\ddot{C}/2C)x^2$, where $C(t)$ is a power function of t, is found to possess an energy-type invariant. More precisely, all potentials investigated using this method culminate in the form

$$V(x, t) = At^{2\lambda-2}\,(x/t^{\lambda})^{2\mu/K} + Bx^2/t^2 \tag{54}$$

where A is arbitrary and K and B are again expressed in terms of λ and μ.

4. CONSTRUCTION OF THE FIRST INVARIANT IN TWO DIMENSIONS: VARIOUS METHODS

As mentioned before, the study of TD systems turns out to be more difficult than the TID systems in a given number of dimensions. For TD systems in 2D, while the available methods seem to be inadequate to provide the second invariant (in order to fulfill the integrability requirement of the system, if it exists), of the system, not many attempts have been made to obtain the invariants of order higher than the second order (in momenta). In this section we present various methods used to study the first invariant of second order in 2D. In particular, the rationalization method and the dynamical algebraic approach will be discussed. Again, in the rationalization method both Cartesian and complex coordinate analysis will be carried out as is done for TID systems by Kaushal *et al.*[44]

4.1. Rationalization Method

4.1.1. Case of Cartesian Coordinates

The direct or rationalization method has been employed by several authors[39,40] to study TD systems in 2D using Cartesian coordinates. For the study of linear (in momenta) invariants we refer to these works. In this section we describe this method in the context of second-order invariants only. For this purpose, we consider a dynamical system described by the Lagrangian

$$L = (1/2)(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1, x_2, t) \tag{55}$$

and for the nth-order invariants, in analogy with (25), one can make an ansatz as

$$\begin{aligned} I_n = a_0 &+ \sum_{i_1=1}^{2} a_{i_1}\xi_{i_1} + (1/2!) \sum_{i_1, i_2=1}^{2} a_{i_1 i_2}\, \xi_{i_1}\, \xi_{i_2} + \ldots \\ &+ (1/n!) \sum_{i_1, i_2 \ldots i_n=1}^{2} a_{i_1 i_2 \ldots i_n}\, \xi_{i_1}\, \xi_{i_2} \ldots \xi_{i_n} \end{aligned} \tag{56}$$

but here for the second-order case we restrict ourselves to the form

$$I = a_0 + a_i\, \xi_i + (1/2!)\, a_{ij}\, \xi_i\, \xi_j \qquad (i, j = 1, 2) \tag{57}$$

where $\xi_i = \dot{x}_i$; $a_{ij} = a_{ji}$ and a_0, a_i, a_{ij} are now functions of x_1, x_2, and t. Using (5) and (6) (with the Poisson bracket now defined for the 2D case) for the invariance of I and after accounting for the proper symmetrization, one arrives[41] at the following relations for the coefficient functions a_{ij}, a_i, and a_0:

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0 \tag{58}$$

$$a_{i,j} + a_{j,i} = -\partial a_{ij}/\partial t \tag{59}$$

$$a_{0,i} + a_{ij}\dot{\xi}_j = -\,\partial a_i/\partial t \tag{60}$$

$$a_i\,\dot{\xi}_i = -\partial a_0/\partial t \tag{61}$$

In their detailed form these PDEs can be written as

$$(\partial a_{11}/\partial x_1) = 0 \tag{62a}$$

$$(\partial a_{22}/\partial x_2) = 0 \tag{62b}$$

$$2(\partial a_{12}/\partial x_1) + (\partial a_{11}/\partial x_2) = 0 \tag{62c}$$

$$2(\partial a_{12}/\partial x_2) + (\partial a_{22}/\partial x_1) = 0 \tag{62d}$$

$$2(\partial a_1/\partial x_1) = -\partial a_{11}/\partial t \tag{62e}$$

$$2(\partial a_2/\partial x_2) = -(\partial a_{22}/\partial t) \tag{62f}$$

$$(\partial a_1/\partial x_2) + (\partial a_2/\partial x_1) = -(\partial a_{12}/\partial t) \tag{62g}$$

$$\partial a_0/\partial x_1 - a_{11}(\partial V/\partial x_1) - a_{12}(\partial V/\partial x_2) = -\,(\partial a_1/\partial t) \tag{62h}$$

$$\partial a_0/\partial x_2 - a_{12}(\partial V/\partial x_1) - a_{22}(\partial V/\partial x_2) = -\,(\partial a_2/\partial t) \tag{62i}$$

$$a_1(\partial V/\partial x_1) + a_2(\partial V/\partial x_2) = (\partial a_0/\partial t) \tag{62j}$$

Note that the presence of the term linear in momenta in (57) leads to a larger number of equations here as compared to that of the TID case. Further, a simple analysis of equations (62a), (62b), (62c), and (62d) immediately leads[41] to the forms of the a_{ij} as

$$\begin{aligned} a_{11}(x_2, t) &= \psi_0(t)x_2^2 + \psi_2(t)x_2 + \psi_3(t) \\ a_{22}(x_1, t) &= \psi_0(t)x_1^2 + \psi_1(t)x_1 + \psi_4(t) \\ a_{12}(x_1, x_2, t) &= -\psi_0(t)x_1x_2 - \tfrac{1}{2}[\psi_2(t)x_1 + \psi_1(t)x_2 - \mu(t)] \end{aligned} \tag{63}$$

and subsequently the integration of equations (62e)–(62g) yields for the a_i the expressions

$$\begin{aligned} a_1(x_1, x_2, t) &= -\tfrac{1}{2}[\dot{\psi}_2(t)x_2 + \dot{\psi}_3(t)]x_1 + \tfrac{1}{2}\dot{\psi}_1(t)x_2^2 \\ &\quad - \tfrac{1}{2}[\dot{\mu}(t) + \psi_5(t)]x_2 + \tfrac{1}{2}\psi_7(t) \\ a_2(x_1, x_2, t) &= -\tfrac{1}{2}[\dot{\psi}_1(t)x_1 + \dot{\psi}_4(t)]x_2 + \tfrac{1}{2}\dot{\psi}_2(t)x_1^2 \\ &\quad + \tfrac{1}{2}\psi_5(t)x_1 + \tfrac{1}{2}\psi_6(t) \end{aligned} \tag{64}$$

Here the ψ_i are arbitrary functions of t. From now onward we drop the arguments of the respective functions for the sake of brevity. Finally, after eliminating a_0 from (62h) and (62i) by differentiating them w.r.t. x_2 and x_1, respectively, and subsequently using the results (63) and the fact that $(\partial^2 a_0/\partial x_1 \partial x_2) = (\partial^2 a_0/\partial x_2\, \partial x_1)$, one arrives[41] at the following "potential" equation for the second-order invariants:

$$
\begin{aligned}
&3[\ddot{\psi}_2 x_2 - \ddot{\psi}_1 x_1 + \tfrac{1}{2}\ddot{\mu} + \dot{\psi}_5] + 3(2\psi_0 x_2 + \psi_2)\,(\partial V/\partial x_1) \\
&\quad -3(2\psi_0 x_1 + \psi_1)(\partial V/\partial x_2) + (2\psi_0 x_1 x_2 + \psi_2 x_1 + \psi_1 x_2 - \mu) \\
&\quad \times [(\partial^2 V/\partial x_1^2) - (\partial^2 V/\partial x_2^2)] + 2(\psi_0(x_2^2 - x_1^2) + \psi_2 x_2 - \psi_1 x_1 + \psi_3 - \psi_4) \\
&\quad \times (\partial^2 V/\partial x_1\, \partial x_2) = 0
\end{aligned}
\tag{65}
$$

This result was also derived by Grammaticos and Dorizzi[40] and was used to study a number of TD systems in 2D. The handling of this equation is even more difficult than the corresponding equation in the TID case.[1,2,41] An equation similar to (65) for the TID case has been studied by Hietarinta[2] and others. What they have actually analyzed are cases corresponding to different values of the coefficients c_i [cf. equation (13) in ref. 41)], whereas we have looked[41] at two major cases corresponding to the separation (under addition and multiplication) of the potential function in x_1 and x_2 variables by keeping the c_i free. In the same spirit we have made an attempt here to solve equation (65) in a general manner by resorting to separable forms of $V(x_1, x_2, t)$ in the variables x_1 and x_2 as before.

The case when

$$V(x_1, x_2, t) = f(x_1, t) + g(x_2, t)$$

which is one of the special solutions of (65), yields the form of V as

$$V(x_1, x_2, t) = \psi_0[x_1 + \psi_1/2\psi_0]^2 + \psi_0[x_2 + \psi_2/2\psi_0]^2 + v_1(t) \tag{66}$$

which, in analogy with the TID case, is the case of a *shifted rotating harmonic oscillator*. Here $v_1(t) = \psi_8 + \psi_0 - (\psi_1^2 + \psi_2^2)/2\psi_0$ is the pure TD part of V. Invariants are also constructed for several other special solutions of (65).

The rationalization of (65) for a number of TD coupled oscillator systems has led to the construction of invariants of several interesting cases. Here we mention only the pertinent results.

(i) *TD Coupled Oscillators*: For the system described by the potential

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta_1(t)x_1x_2 \tag{67}$$

the invariant is obtained[41] as

$$I = (\alpha_1\psi_3 + \tfrac{1}{4}\ddot{\psi}_3)x_1^2 + (\alpha_2\psi_3 + \tfrac{1}{4}\ddot{\psi}_3)x_2^2 + \beta_1\psi_3 x_1 x_2$$
$$+ \tfrac{1}{2}(\dot{\psi}_7 x_1 - \dot{\psi}_6\, x_2) + \tfrac{1}{2}(\psi_7\dot{x}_1 + \psi_6\dot{x}_2) - \tfrac{1}{2}\dot{\psi}_3\,(x_1\dot{x}_1 + x_2\dot{x}_2)$$
$$+ \tfrac{1}{2}c_5(x_1\dot{x}_2 - \dot{x}_1 x_2) + \tfrac{1}{2}\psi_3(\dot{x}_1^2 + \dot{x}_2^2) \tag{68}$$

where

$$\psi_3(t) = [c_5/2(\alpha_1 - \alpha_2)^{1/2}] \int [\beta_1/(\alpha_1 - \alpha_2)^{1/2}]\, dt$$

with $c_5 \equiv$ const; $\psi_6(t)$ and $\psi_7(t)$ are give by

$$\ddot{\psi}_6 = -2\psi_6\alpha_2 - \psi_7\beta_1; \qquad \ddot{\psi}_7 = 2\psi_7\alpha_1 + \psi_6\beta_1$$

and α_1, α_2 and β_1 satisfying the constraint

$$[\dot{\beta}_1(\alpha_1 - \alpha_2) - \beta_1(\dot{\alpha}_1 - \dot{\alpha}_2)]\, B_1 + 2(\alpha_1 - \alpha_2)^{1/2}[\beta_1^2 + (\alpha_1$$
$$- \alpha_2)^2] = 0 \tag{69}$$

with $B_1 = \int (\beta_1/(\alpha_1 - \alpha_2)^{1/2})\, dt.$

(ii) *TD Oscillator with Inverse Harmornic and Cross Terms*: In this case, we consider a general system described by the potential

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta(t)x_1^m x_2^n \tag{70}$$

and notice that the rationalization of the potential equation (65) is possible only for the choice of the arbitrary functions $\psi_0 = \psi_1 = \psi_2 = \mu = 0$, $\psi_3 = \psi_4$. Further, $\ddot{\psi}_5 = 0$, implying $\psi_5 =$ const (say c_5) and the numbers m and n must satisfy $m + n = -2$, as we obtained for the TID case.[41] Though with this restriction on m and n several choices (including their fractional values) are possible, we give here results only for a few cases.

For the potential corresponding to $m = n = -1$, viz.,

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta_0 x_1^{-1} x_2^{-1} \tag{71}$$

the invariant turns out to be

$$I = (\alpha_1\psi_3 + \tfrac{1}{4}\ddot{\psi}_3)x_1^2 + (\alpha_2\psi_3 + \tfrac{1}{4}\ddot{\psi}_3)x_2^2 + \beta_0\psi_3 x_1^{-1} x_2^{-1}$$
$$- \tfrac{1}{2}\dot{\psi}_3(x_1\dot{x}_1 + x_2\dot{x}_2) + \tfrac{1}{2}\psi_3(\dot{x}_1^2 + \dot{x}_2^2 \tag{72}$$

where ψ_3, α_1, and α_2 satisfy the equations

$$\tfrac{1}{4}\dddot{\psi}_3 + 2\alpha_1\dot{\psi}_3 + \dot{\alpha}_1\psi_3 = 0; \qquad \tfrac{1}{4}\dddot{\psi}_3 + 2\alpha_2\dot{\psi}_3 + \dot{\alpha}_2\psi_3 = 0$$

and $c_5 = \psi_6 = \psi_7 = 0$. The above equations for ψ_3 yield $\psi_3(t) = c_3(\alpha_1 - \alpha_2)^{-1/2}$, and a constraint on α_1 and α_2 as

$$4(\ddot{\alpha}_1 - \ddot{\alpha}_2)(\alpha_1 - \alpha_2)^2 - 18(\alpha_1 - \alpha_2)\,(\dot{\alpha}_1 - \dot{\alpha}_2)(\ddot{\alpha}_1 - \ddot{\alpha}_2) + 15(\dot{\alpha}_1 - \dot{\alpha}_2)^3$$
$$= 32(\alpha_1 - \alpha_2)^3(\alpha_1\dot{\alpha}_2 - \dot{\alpha}_1\alpha_2) \tag{73}$$

Further, a symmetrization of the $x_1^m x_2^n$ term in (70) with respect to m and n, namely the replacement of $\beta(t)x^m x^n$ by $(\beta_1(t)x_1^m x_2^n + \beta_2(t)x_1^n x_2^m)$ leads to several new results. For example, for $m = 0$, $n = 2$ and $m = 2$, $n = 0$ or for $m = 1$, $n = -3$ and $m = -3$, $n = 1$ the invariants corresponding to the TD oscillator with inverse harmonic or with cross terms can easily be derived.[41] Not only this, but a criterion for the relative time dependence of α_1, α_2, and β can be set. We shall return to some of these discussions in the context of the dynamical algebraic approach.

4.1.2. Case of Complex Coordinates

The complexification of two space dimensions in the form $Z = x_1 + ix_2$, $\bar{Z} = x_1 - ix_2$ for TD systems has also led to some interesting results as far as the construction of the first invariant is concerned. In particular, the integrability of a variety of central potentials in this approach can be confirmed rather easily as compared to that in the Cartesian case. The integrability of TID systems in 2D has also been studied[60] within this framework. Here we extend these results to the case of TD systems and restrict ourselves to the construction of second-order invariants in the ansatz (56). The systems we consider now are described by the Lagrangian

$$L = (1/2)|\dot{Z}|^2 - V(Z, \bar{Z}, t) \tag{74}$$

with the corresponding equations of motion given by

$$\ddot{Z} = -2(\partial V/\partial \bar{Z}); \qquad \ddot{\bar{Z}} = -2(\partial V/\partial Z) \tag{75}$$

We make[42] the same ansatz for I as (57), but now $\xi_1 = \dot{Z}$, $\xi_2 = \dot{\bar{Z}}$ and a_0, a_i, a_{ij} are functions of Z, $\bar{Z}$, and t. Of course, the forms of the equations satisfied by a_{ij}, a_i, and a_0 remain the same as (58)–(61), but their detailed versions now turn out to be [42]

$$(\partial a_{11}/\partial Z) = 0 \tag{76a}$$

$$(\partial a_{22}/\partial \bar{Z} = 0 \tag{76b}$$

$$2(\partial a_{12}/\partial Z) + (\partial a_{11}/\partial \bar{Z}) = 0 \tag{76c}$$

$$2(\partial a_{12}/\partial \bar{Z}) + (\partial a_{22}/\partial Z) = 0 \tag{76d}$$

$$2(\partial a_1/\partial Z) = -(\partial a_{11}/\partial t) \tag{76e}$$

$$2(\partial a_2/\partial \bar{Z}) = -(\partial a_{22}/\partial t) \tag{76f}$$

$$(\partial a_1/\partial \bar{Z}) + (\partial a_2/\partial Z) = -(\partial a_{12}/\partial t \tag{76g}$$

$$(\partial a_0/\partial Z) - 2a_{11}(\partial V/\partial \bar{Z}) - 2a_{12}(\partial V/\partial Z) = -(\partial a_1/\partial t) \tag{76h}$$

$$(\partial a_0/\partial \bar{Z}) - 2a_{12}(\partial V/\partial \bar{Z}) - 2a_{22}(\partial V/\partial Z) = -(\partial a_2/\partial t) \tag{76i}$$

$$2a_1(\partial V/\partial \bar{Z}) + 2a_2(\partial V/\partial Z) = (\partial a_0/\partial t) \tag{76j}$$

As before, the solution of these PDEs will lead[42,43] to the following expressions for the coefficient functions a's:

$$a_{11} = c_1\bar{Z}^2 + \psi_2\bar{Z} + \psi_3 \tag{77a}$$

$$a_{22} = c_1Z^2 + \psi_1 Z + \psi_4 \tag{77b}$$

$$a_{12} = -c_1 Z\bar{Z} - \tfrac{1}{2}\psi_2 Z - \tfrac{1}{2}\bar{Z} + \tfrac{1}{2}\mu \tag{77c}$$

$$a_1 = -\tfrac{1}{2}[\dot{\psi}_2\bar{Z} + \dot{\psi}_3]Z + \tfrac{1}{2}\dot{\psi}_1\bar{Z}^2 - \tfrac{1}{2}[\dot{\mu} + \psi_5]\bar{Z} + \psi_7/2 \tag{77d}$$

$$a_2 = -\tfrac{1}{2}[\dot{\psi}_1 Z + \dot{\psi}_4]\bar{Z} + \tfrac{1}{2}\dot{\psi}_2 Z^2 + \tfrac{1}{2}\psi_5 Z + \psi_6/2 \tag{77e}$$

where ψ_i ($i = 1, \ldots, 7$) and μ are arbitrary functions of t, and c_1 is in general a complex separation constant.

Now differentiating (76h) w.r.t. $\bar{Z}$ and (76i) w.r.t. Z and using $(\partial^2 a_0/\partial Z\partial\bar{Z}) = (\partial^2 a_0/\partial\bar{Z}\partial Z)$ and $(\partial^2 V/\partial Z\partial\bar{Z}) = (\partial^2 V/\partial\bar{Z}\partial Z)$, one arrives at

$$\begin{aligned} &a_{22}(\partial^2 V/\partial Z^2) + [(\partial a_{22}/\partial Z) - (\partial a_{12}/\partial\bar{Z})](\partial V/\partial Z) - a_{11}\,(\partial^2 V/\partial\bar{Z}^2) \\ &\quad + [(\partial a_{12}/\partial Z) - (\partial a_{11}/\partial\bar{Z})]\,(\partial V/\partial\bar{Z}) + \tfrac{1}{2}[(\partial^2 a_1/\partial t\partial\bar{Z}) - (\partial^2 a_2/\partial t\partial Z)] = 0 \end{aligned} \tag{78}$$

Similarly, using $(\partial^2 a_0/\partial Z\cdot\partial t) = (\partial^2 a_0/\partial t\cdot\partial Z)$ and $(\partial^2 a_0/\partial\bar{Z}\cdot\partial t) = (\partial^2 a_0/\partial t\cdot\partial\bar{Z})$ in (76h) and (76j), (76i), and (76j), respectively, one obtains another pair of equations,

$$\begin{aligned} &a_2(\partial^2 V/\partial Z^2) + [(\partial a_2/\partial Z) - (\partial a_{12}/\partial t)]\,(\partial V/\partial Z) + a_1(\partial^2 V/\partial\bar{Z}\partial Z) \\ &\quad + [(\partial a_1/\partial Z) - (\partial a_{11}/\partial t)](\partial V/\partial\bar{Z}) - a_{11}(\partial^2 V/\partial t\partial\bar{Z}) - a_{12}(\partial^2 V/\partial t\partial Z) \\ &\quad + \tfrac{1}{2}(\partial^2 a_1/\partial t^2) = 0 \end{aligned} \tag{79}$$

$$a_1(\partial^2 V/\partial\bar{Z}^2) + [(\partial a_1/\partial\bar{Z}) - (\partial a_{12}/\partial t)]\,(\partial V/\partial\bar{Z}) + a_2\,(\partial^2 V/\partial Z\partial\bar{Z})$$

$$+ [(\partial a_2/\partial \bar{Z}) - (\partial a_{22}/\partial t)](\partial V/\partial Z) - a_{22}\,(\partial^2 V/\partial t\partial Z) - a_{12}(\partial^2 V/\partial t\partial \bar{Z})$$
$$+ \tfrac{1}{2}(\partial^2 a_2/\partial t^2) = 0 \tag{80}$$

In a way, (78)–(80) may constitute "potential" equations for the TD case. For a given form of V, while the rationalization of these three equations will fix some arbitrary functions involved in (77), the rest can be determined in the process of computing a_0 from (76j). Further note that for most of systems, (79) and (80) provide identical results.

For the choice $\ddot{\mu}(t) = 0$, $\dot{\psi}_5\,(t) = 0$, $\psi_1 = \psi_2 = \bar{\psi}_2$, and $\psi_3 = \psi_4 = \bar{\psi}_4$, (78) yields[44,45]

$$A(\partial^2 V/\partial Z^2) + B(\partial V/\partial Z) + C = \bar{A}\,(\partial^2 V/\partial \bar{Z}^2) + \bar{B}\,(\partial V/\partial \bar{Z}) + \bar{C}$$
$$\equiv \phi(t) \quad \text{(say)} \tag{81}$$

where $A = 2(c_1 Z^2 + \psi_1 Z + \psi_3)$, $B = 3(2c_1 Z + \psi_1)$, $C = -\,(3/2)\,\ddot{\psi}_2 Z$. If $V(Z, \bar{Z}, t) = V(|Z|, t) = \beta(t)v(|Z|)$, then the invariants are constructed[43] for the forms $V(|Z|, t) = \beta(t)(b/r^4 + d)$ (van der Waals-type potential) and $V(|Z|, t) = \beta(t)\,(\ln r + b_1/r^4 + d_1)$ (confining-type potential). Other TD central potentials investigated are (i) the linear confining potential

$$V(|Z|, t) = \omega(t)(Z\bar{Z})^{1/2} - \beta(t)(Z\bar{Z})^{-1/2} \tag{82}$$

for which the invariant turns out to be[42]

$$I = \mu^{-1/2}(r - \mu/r) - (1/2)\dot{\mu}(x_1\dot{x}_1 + x_2\dot{x}_2) - 2c_1\,(x_1\dot{x}_2 - x_2\dot{x}_1)^2$$
$$+ (1/2)\mu(\dot{x}_1^2 + \dot{x}_2^2) \tag{83}$$

where $r^2 = Z\bar{Z}$ and $\mu = at^2 + b't + c'$, and (ii) the harmonic confining potential

$$V(|Z|, t) = -\,(1/2)\,(\ddot{u}/u)Z\bar{Z} - (\mu_0/u)(Z\bar{Z})^{-1/2} \tag{84}$$

which was also studied by Katzin and Levine[46] using the method of symmetries.[47] The invariant obtained for this latter potential is given by

$$I = [(\dot{\mu}/8\mu)r^2 - (\sqrt{\mu}/\mu_0)r^{-1}]$$
$$+ k\mu_0(x_1\dot{x}_2 - x_2\dot{x}_1)[(\dot{\mu}/\sqrt{\mu})\,x_2 - 2\sqrt{\mu}\dot{x}_2]$$
$$-(1/2)\dot{\mu}(x_1\dot{x}_1 + x_2\dot{x}_2) - 2c_1(x_1\dot{x}_2 - x_2\dot{x}_1)^2$$
$$+ (1/2)\mu(\dot{x}_1^2 + \dot{x}_2^2) \tag{85}$$

where $\mu = u^2/\mu_0^2$; $\psi_1 = \psi_2 = ku$ and $\psi_5 = -(u\dot{u}/\mu_0^2)$. The invariant is also constructed for the TD Kepler potential, $V(|Z|, t) = -\beta(t)(Z\bar{Z})^{-1/2}$.

It is not difficult to extend to two dimensions the method of Lewis and Leach,[39] which is based on an infinite series expansion of I in powers of the momentum. Further, correspondence of this method with the present one can be established[44] by writing the Poisson bracket of I and H as

$$[I, H]_{\mathrm{PB}} = (\partial I/\partial Z)(\partial H/\partial \dot{Z}) - (\partial I/\partial \dot{Z})(\partial H/\partial Z) + (\partial I/\partial \bar{Z})(\partial H/\partial \dot{\bar{Z}}) - (\partial I/\partial \dot{\bar{Z}})(\partial H/\partial \bar{Z})$$

4.2. Dynamical Algebraic Approach

With a view to demonstrating the underlying elegance of the dynamical algebraic approach at the classical level, we present here its extended version to the 2D case, which has been carried out recently by Kaushal and Mishra.[48] No doubt the central idea of the method still remains the same, but now the complexity of the algebra in terms of closure increases enormously.

The Hamiltonian for a 2D system, as before, can be expressed as

$$H = \sum_n h_n(t)\Gamma_n(x_1, p_1, x_2, p_2) \tag{86}$$

where the phase space functions Γ_n are functions of x_1, p_1, x_2, and p_2 and they still close the Lie algebra through (33), but with respect to the Poisson bracket now defined as

$$[f, g]_{\mathrm{PB}} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial x_2} \tag{87}$$

The invariant I, also a member of the dynamical algebra, is now expressed as

$$I = \sum_k \lambda_k(t)\Gamma_k(x_1, p_1, x_2, p_2) \tag{88}$$

and fulfills the requirement (35), which finally leads to a set of equations similar to (36) for determining the unknown λ_k. Here, we employ this method first for the simple case of coupled TD oscillators (67) and then for the case of its generalized version.

4.2.1. Coupled TD Oscillators

The Hamiltonian corresponding to the potential (67) can be written as

$$H = (1/2)(p_1^2 + p_2^2) + \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta(t)x_1x_2 \tag{89}$$

which we wish to express in the form (86) by identifying

$$\Gamma_1 = p_1^2/2, \qquad \Gamma_2 = p_2^2/2, \qquad \Gamma_3 = x_1^2, \qquad \Gamma_4 = x_2^2, \qquad \Gamma_5 = x_1 x_2 \tag{90a}$$

and

$$h_1 = h_2 = 1, \qquad h_3 = \alpha_1(t), \qquad h_4 = \alpha_2(t), \qquad h_5 = \beta(t) \tag{90b}$$

In order to close the algebra, in this case it becomes necessary to introduce five more Γ_k, namely

$$\begin{aligned} \Gamma_6 &= -2p_1x_1, \qquad \Gamma_7 = -p_1x_2, \\ \Gamma_8 &= -2p_2x_2, \qquad \Gamma_9 = -p_2x_1, \\ \Gamma_{10} &= p_1p_2 \end{aligned} \tag{90c}$$

with the corresponding $h_k(t) = 0$ in (86). Further, the number of nonvanishing Poisson brackets turns out[48] to be 28 and their use in (35) yields[21] the following set of first-order ODEs in the λ's:

$$\dot\lambda_1 = 4\lambda_6 \tag{91a}$$

$$\dot\lambda_2 = 4\lambda_8 \tag{91b}$$

$$\dot\lambda_3 = -4\alpha_1\lambda_6 - \beta\lambda_9 \tag{91c}$$

$$\dot\lambda_4 = -\beta\lambda_7 - 4\alpha_2\lambda_8 \tag{91d}$$

$$\dot\lambda_5 = -2\beta\lambda_6 - 2\alpha_1\lambda_7 - 2\beta\lambda_8 - 2\alpha_2\lambda_9 \tag{91e}$$

$$\dot\lambda_6 = -\alpha_1\lambda_1 + \lambda_3 - (1/2)\beta\lambda_{10} \tag{91f}$$

$$\dot\lambda_7 = -\beta\lambda_1 + \lambda_5 - 2\alpha_2\lambda_{10} \tag{91g}$$

$$\dot\lambda_8 = -\alpha_2\lambda_2 + \lambda_4 - (1/2)\beta\lambda_{10} \tag{91h}$$

$$\dot\lambda_9 = -\beta\lambda_2 + \lambda_5 - 2\alpha_1\lambda_{10} \tag{91i}$$

$$\dot\lambda_{10} = \lambda_7 + \lambda_9 \tag{91j}$$

As such the solution of these 10 coupled equations is difficult, but if we set $\lambda_{10} = \text{const}$ (say k), $\lambda_1 = \lambda_2 = \psi(t)$, and $\lambda_7 = -\lambda_4 = \eta(t)$ (say), then (91) can be solved immediately. As a result, the invariant (88) for the system (89) takes the form[48]

$$\begin{aligned} I = {} & (1/2)\psi(p_1^2 + p_2^2) + [(1/4)\ddot\psi + \alpha_1\psi + (1/2)k\beta]x_1^2 \\ & \times [(1/4)\ddot\psi + \alpha_2\psi - (1/2)k\beta]x_2^2 + (\beta\psi + k\alpha_1 + k\alpha_2)x_1x_2 \end{aligned}$$

$$- (1/2)\psi(x_1p_1 + x_2p_2) + \eta(x_1p_2 - x_2p_1) - kp_1p_2 \tag{92}$$

where ψ, η, α_1, α_2, and β satisfy the relations

$$2\beta\dot{\psi} + \dot{\beta}\psi + k(\dot{\alpha}_1 + \dot{\alpha}_2) = -2(\alpha_1 - \alpha_2)\eta; \qquad \dot{\eta} = k(\alpha_1 - \alpha_2) \tag{93}$$

Note that when $k = 0$ (or $\eta =$ const) the result (93) reduces to (68), but only after setting $\psi_6 = \psi_7 = 0$ in the latter.

4.2.2. Generalized TD Oscillators

Now we consider a generalized form of (89) in which the coupling term $\beta(t)x_1x_2$ is replaced by an arbitrary function $\beta(t)\phi$, namely

$$\begin{aligned} H &= (1/2)(p_1^2 + p_2^2) + \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta(t)\phi(x_1, x_2) \\ &\equiv \Gamma_1 + \Gamma_2 + \alpha_1(t)\Gamma_3 + \alpha_2(t)\Gamma_4 + \beta(t)\Gamma_5 \end{aligned} \tag{94}$$

with Γ_1, Γ_2, Γ_3, and Γ_4 as defined before [cf. equation (90a)] and $\Gamma_5 = \phi(x_1, x_2)$. As a result of this new definition of Γ_5 the affected (nonvanishing) Poisson brackets can be computed as[48]

$$\begin{aligned} &[\Gamma_1, \Gamma_5]_{PB} = -p_1\frac{\partial\phi}{\partial x_1}; \qquad [\Gamma_2, \Gamma_5]_{PB} = -p_2\frac{\partial\phi}{\partial x_2} \\ &[\Gamma_5, \Gamma_6]_{PB} = -2x_1\frac{\partial\phi}{\partial x_1}; \qquad [\Gamma_5, \Gamma_8]_{PB} = -2x_2\frac{\partial\phi}{\partial x_2} \end{aligned} \tag{95}$$

In this case, Γ_7 and Γ_9 are absent from the Lie algebra and Γ_{10} defined in the earlier case from $[\Gamma_2,\Gamma_7]_{PB} = \Gamma_{10}$ is also absent. Finally, one is left here only with seven coupled equations in the λ's, namely

$$\dot{\lambda}_1 = 4\lambda_6 \tag{96a}$$

$$\dot{\lambda}_2 = 4\lambda_8 \tag{96b}$$

$$\dot{\lambda}_3 = -4\alpha_1\lambda_6 \tag{96c}$$

$$\dot{\lambda}_4 = -4\alpha_2\lambda_8 \tag{96d}$$

$$\begin{aligned} \dot{\lambda}_5\phi &= (\beta\lambda_1 - \lambda_5)p_1\frac{\partial\phi}{\partial x_1} + (\beta\lambda_2 - \lambda_5)p_2\frac{\partial\phi}{\partial x_2} - 2\beta\lambda_6x_1\frac{\partial\phi}{\partial x_1} \\ &\quad - 2\beta\lambda_8x_2\frac{\partial\phi}{\partial x_2} \end{aligned} \tag{96e}$$

$$\dot{\lambda}_6 = -\alpha_1\lambda_1 + \lambda_3 \tag{96f}$$

$$\dot{\lambda}_8 = -\alpha_2\lambda_2 + \lambda_4 \tag{96g}$$

As before, here we again make the ansatz $\lambda_1 = \lambda_2 = \psi(t)$, and obtain the solution for other λ's from (96). This gives rise to the constraining relations

$$(\ddot{\psi}/4) + 2\alpha_1\dot{\psi} + \dot{\alpha}_1\psi = 0 \tag{97a}$$

$$(\ddot{\psi}/4) + 2\alpha_2\dot{\psi} + \dot{\alpha}_2\psi = 0 \tag{97b}$$

and the form of the ϕ equation (96e) as

$$\dot{\lambda}_5\phi - (\beta\psi - \lambda_5)\left(p_1\frac{\partial\phi}{\partial x_1} + p_2\frac{\partial\phi}{\partial x_2}\right) + \frac{1}{2}\beta\dot{\psi}\left(x_1\frac{\partial\phi}{\partial x_1} + x_2\frac{\partial\phi}{\partial x_2}\right) = 0 \tag{98}$$

For the case when

$$\lambda_5 = \beta\psi \tag{99}$$

two particular solutions of (98) (namely, the ones separable in x_1 and x_2 coordinates under addition and multiplication operations) lead[48] to interesting cases:

(i)

$$\phi(x_1, x_2) = k_1x_1^{-\delta} + k_2x_2^{-\delta} \tag{100a}$$

(ii)

$$\phi(x_1, x_2) = k_3(x_1/x_2)^{c_1}x_1^{-\delta} + k_4(x_2/x_1)^{c_1}x_2^{-\delta} \tag{100b}$$

Here, c_1 and k_i ($i = 1, 2, 3, 4$) are the separation and integration constants, respectively, and the function $\delta(t)$ is given by

$$\delta(t) = 2 + \dot{\beta}\psi/(\beta\dot{\psi}) \tag{101a}$$

which, after using the form $\psi = c_0(\alpha_1 - \alpha_2)^{-1/2}$ [cf. equations' (97)], reduces to

$$\delta(t) = 2 - \dot{\beta}(\alpha_1 - \alpha_2)/\beta(\dot{\alpha}_1 - \dot{\alpha}_2) \tag{101b}$$

Now, it is not difficult to write down the invariants for the systems corresponding to the cases (100a) and (100b), which, respectively, turn out to be

$$I = \tfrac{1}{2}\psi(p_1^2 + p_2^2) + (\tfrac{1}{4}\ddot{\psi} + \alpha_1\psi)x_1^2 + (\tfrac{1}{4}\ddot{\psi} + \alpha_2\psi)x_2^2$$
$$+ \beta\psi(k_1 x_1^{-\delta} + k_2 x_2^{-\delta}) - \tfrac{1}{2}\mathring{\psi}(x_1 p_1 + x_2 p_2)$$

and

$$I = \tfrac{1}{2}\psi(p_1^2 + p_2^2) + (\tfrac{1}{4}\ddot{\psi} + \alpha_1\psi)x_1^2 + (\tfrac{1}{4}\ddot{\psi} + \alpha_2\psi)x_2^2$$
$$+ \beta\psi[k_3(x_1/x_2)^{c_1}x_1^{-\delta} + k_4(x_2/x_1)^{c_1}x_2^{-\delta}] - \tfrac{1}{2}\dot{\psi}(x_1 p_1 + x_2 p_2)$$

No doubt the solution of the PDE (98) can suggest further examples of systems admitting the first invariant, but what is of importance here is the rationale suggested by the present approach in terms of this equation regarding the relative time dependence of the couplings in (94) vis-à-vis the time dependence of the exponent δ. In this connection the following remarks are in order:

Equation (101b) implies that the time dependence in $\delta(t)$ arises mainly from the fact that $\alpha_1(t) \neq \alpha_2(t)$ in (94), i.e., for the oscillators with unequal spring constants. Alternatively, if $\dot{\beta}\psi = 0$ in (101a), then δ becomes independent of t and attains the value $\delta = 2$. As a result, since $\psi \neq 0$, $\dot{\beta}$ must be zero, thereby implying $\beta(t) = \text{const}$ (say β_0). This will lead to the systems

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta_{10}\, x_1^{-2} + \beta_{20} x_2^{-2} \tag{102}$$

and

$$V(x_1, x_2, t) = \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 + \beta_{10}\,(x_1/x_2)^{c_1}x_1^{-2} + \beta_{20}(x_2/x_1)^{c_1}x_2^{-2} \tag{103}$$

for which the invariants can also be constructed using the rationalization method. In fact, the system (103) is of special interest from the point of view of generalizing the Ermakov systems (cf. Section 6). By writing the cross term in (103) as $\phi = \beta_{10}\, x_1^m x_2^n + \beta_{20} x_1^n\, x_2^m$, one can notice that $m + n = -2$. This is another important rationale for choosing the coupling terms in the systems admitting quadratic invariants, and valid for both TD and TID[41] systems.

4.3. Painlevé Method

There are not many applications of the Painlevé conjecture (see, for example, ref. 49) to study TD systems in 2D. However, the applications seem to be straightforward and can be carried out more or less in the same manner as is done for TID systems.[5]

5. THIRD- AND HIGHER ORDER INVARIANTS

In view of the fact that there exist additional difficulties in dealing with systems involving an explicit time dependence, not many attempts have been made to construct their invariants of order higher than two. Here, we present a brief derivation of some results in the form of PDEs whose solutions would directly provide systems admitting the third- or fourth-order invariants in both one and two dimensions. No doubt the method of Feix *et al.*,[38] based on the use of self-similar techniques, also provides invariants of higher order in momenta, but it seems that only a restricted class of potentials (cf. Section 3.6) can be studied using this method. Therefore, we continue with the more general method, i.e., the rationalization method.

5.1. Higher Order Invariants in One Dimension

We consider the system (26) and use the ansatz (25) along with the recursion relation (27).

5.1.1. Third-Order Invariants

For this case, in addition to (27c) and (27d), we also have[29] other PDEs from (27) as

$$(\partial b_3/\partial x) = 0 \tag{104a}$$

$$3(\partial b_2/\partial x) + (\partial b_3/\partial t) = 0 \tag{104b}$$

$$2(\partial b_1/\partial x) + (\partial b_2/\partial t) - b_3(\partial V/\partial x) = 0 \tag{104c}$$

The first two equations, respectively, provide

$$b_3 = \psi_1(t), \qquad b_2 = -(1/3)\dot{\psi}_1 x + \psi_2(t) \tag{105}$$

Substituting these results for b_3 and b_2 in (104c) and integrating the resultant equation, one obtains

$$b_1 = (1/12)\ddot{\psi}_1 x^2 - (1/2)\dot{\psi}_1 x + (1/2)\psi_1 V + \psi_3(t) \tag{105$'$}$$

where the ψ_i are arbitrary functions of t. Using these results for b_1 and b_2, (27c) and (27d) can be used to eliminate b_0 in favor of V by noting that $(\partial^2 b_0/\partial x\,\partial t) = (\partial^2 b_0/\partial t\,\partial x)$. This will lead to a general "potential" equation for the third-order invariants

$$\begin{aligned}
&\tfrac{1}{2}\psi_1 V(\partial^2 V/\partial x^2) + [(1/12)\ddot{\psi}_1 x^2 - \tfrac{1}{2}\dot{\psi}_2 x\psi_3](\partial^2 V/\partial x^2) \\
&\quad + [\tfrac{1}{2}\psi_1(\partial V/\partial x)^2 + \tfrac{1}{2}(\ddot{\psi}_1 x - 3\dot{\psi}_2)](\partial V/\partial x) + (\tfrac{1}{3}\dot{\psi}_1 x - \psi_2)
\end{aligned}$$

$$\times (\partial^2 V/\partial x \cdot \partial t) + \tfrac{1}{2}\psi_1(\partial^2 V/\partial t^2) + \dot{\psi}_1(\partial V/\partial t) + \tfrac{1}{2}\ddot{\psi}_1 V$$

$$+ [(1/12)\ddddot{\psi}_1 x^2 - \tfrac{1}{2}\dddot{\psi}_2 x - \ddot{\psi}_3] = 0 \tag{106}$$

Clearly the solution of this nonlinear equation is difficult. However, we notice from (105′) that $\psi_1(t) = 2(\partial^3 b_1/\partial x^3)/(\partial^3 V/\partial x^3)$, which implies the separability of b_1 and also of V in x and t variables to the extent that the ratio $(\partial^3 b_1/\partial x^3)/(\partial^3 V/\partial x^3)$ is only a function of t. For this reason we choose

$$b_1(x, t) = f(t)v(t), \qquad V(x, t) = (2f(t)/\psi_1)v(t) \tag{107}$$

Such a choice would reduce[29] the potential equation (106) to a simpler form

$$(1/v)(d(v dv/dx)/dx) + [(\dot{f}\psi_1 - c_3 f)(\tfrac{1}{3}c_3 x - c_4)/\psi_1 f^2]$$

$$\times (1/v)(dv/dx) + (\psi_1 \ddot{f}/2f^2) = 0 \tag{108}$$

where c_3 and c_4 are arbitrary constants of integration. Finally, an integrable system of the form

$$V(x, t) = a t^{-4/3}(k_4 x + k_5)^{1/2} \tag{109}$$

admitting the invariant

$$I = -(1/2)[f - (k_4 x + k_5)^{1/2}\dot{x}]^2 + (1/6)\psi_1 \dot{x}^3 \tag{110}$$

is obtained.[29] Here $f(t) \sim t^{-1/3}$ and $\psi_1(t) \sim t$.

5.1.2. Fourth-Order Invariants

In this case, in addition to (27c), (27d), and (104c), from (27) we also have

$$(\partial b_4/\partial x) = 0 \tag{111a}$$

$$4(\partial b_3/\partial x) + (\partial b_4/\partial t) = 0 \tag{111b}$$

$$3(\partial b_2/\partial x) + (\partial b_3/\partial t) - b_4(\partial V/\partial x) = 0 \tag{111c}$$

As before, (111a) and (111b) lead to

$$b_4 = s_1(t) \tag{112a}$$

$$b_3 = -(1/4)\dot{s}_1 x + s_2(t) \tag{112b}$$

which in turn provide the solution of (111c) as

$$b_2 = (1/24)\ddot{s}_1 x^2 - (1/3)\dot{s}_2 x + (1/3)s_1 V + s_3(t) \tag{112c}$$

where the s_i are some arbitrary functions of t. As before, using these expressions for b_2, b_3, and b_4, the integration of (104c) yields

$$b_1 = (-1/144)\dddot{s}_1 x^3 + (1/12)\ddot{s}_2 x^2$$
$$-(1/24)\dot{s}_1 \mathcal{V} - (1/6)s_1(\partial\mathcal{V}/\partial t)$$
$$-(1/2)\dot{s}_3 x - (1/2)[(1/4)\,\dot{s}_1 x - s_2]V + s_4(t) \tag{112d}$$

where $\mathcal{V} = \int V\,dx$ and s_4 is another arbitrary function of t. As for the third-order case, b_0 from (27c) and (27d) can be eliminated in favor of V again by noting that $(\partial^2 b_0/\partial x \cdot \partial t) = (\partial^2 b_0/\partial t \cdot \partial x)$. This will, in fact, determine the "potential" equation for the fourth-order invariants.

$$[(1/144)s_1^{(3)}x^3 - (1/12)s_2^{(2)}x^2 + (1/2)\dot{s}_3 x - s_4 + (1/24)\dot{s}_1\mathcal{V}$$
$$+ \tfrac{1}{6}s_1\,(\partial\mathcal{V}/\partial t) + \tfrac{1}{2}\,(\tfrac{1}{4}\dot{s}_1 x - s_2)V]\,(\partial^2 V/\partial x^2)$$
$$+ \tfrac{1}{2}\,(\tfrac{1}{4}\dot{s}_1 x - s_2)(\partial V/\partial x)^2 + \tfrac{1}{2}\,[\tfrac{1}{8}s_1^{(3)}x^2 - s_2^{(2)}x$$
$$+ 3\dot{s}_3 + \dot{s}_1 V + s_1(\partial V/\partial t)]\,(\partial V/\partial x) + [(1/24)s_1^{(2)}x^2 - \tfrac{1}{3}\dot{s}_2 x$$
$$+ s_3 + \tfrac{1}{3}s_1 V]\,(\partial^2 V/\partial x \cdot \partial t) + [(1/144)s_1^{(5)}x^3 - (1/12)s_2^{(4)}x^2$$
$$+ \tfrac{1}{2}\,s_3^{(3)}x - s_4^{(2)} + (1/24)\,s_1^{(3)}\mathcal{V} + \tfrac{1}{4}s_1^{(2)}(\partial\mathcal{V}/\partial t)$$
$$+ \tfrac{3}{8}\dot{s}_1(\partial^2\mathcal{V}/\partial t^2) + \tfrac{1}{6}s_1(\partial^3\mathcal{V}/\partial t^3) + \tfrac{1}{2}\,(\tfrac{1}{4}s_1^{(3)}x - s_2^{(2)})V$$
$$+ (\tfrac{1}{2}\,s_1^{(2)}x - \dot{s}_2)(\partial V/\partial t) + \tfrac{1}{2}\,(\tfrac{1}{4}\dot{s}_1\,x - s_2)(\partial^2 V/\partial t^2)] = 0 \tag{113}$$

where the superscript numbers in parentheses of the s_i represent the order of time derivatives of the s_i. Note that the potential equation (113) is a nonlinear, integro-PDE whose solution in principle would directly provide the integrable systems admitting fourth-order invariants.

As before, by writing $b_2(x, t)$ and $V(x, t)$ as separable functions in x and t variables as

$$b_2(x, t) = g(t)w(x), \qquad V(x, t) = (3g(t)/s_1)w(x) \tag{114}$$

the potential equation (113) can be expressed in a reduced form

$$(\dot{g}s_1 - \tfrac{3}{4}g\dot{s}_1)Ww'' + 3g\,(\tfrac{1}{4}\ddot{s}_1 x - s_2)(ww')'$$
$$+ (5\dot{g}s_1 - 2g\dot{s}_1)\;ww' + (1/g)(\tfrac{1}{3}\dddot{g}s_1 - \tfrac{1}{4}\ddot{g}\dot{s}_1 s_2$$
$$+ \tfrac{1}{2}\,\dot{g}\dot{s}_1^2 - \dot{s}_1^3 g/2s_1)W + (1/g)(\tfrac{1}{4}\dot{s}_1 x - s_2)$$
$$(\ddot{g}s_1 - 2\dot{s}_1\dot{g} + 2g\dot{s}_1^2/s_1)\;w = 0 \tag{115}$$

where $W = \int w\;dx$ and $s_4\,(t)$ is taken to be zero, and the primes on w represent the derivatives w.r.t. x. Equation (115) appears to be difficult even for a trivial case like the one discussed in the third-order case.

5.2 Third-Order Invariants in Two Dimensions

As the complexity in the derivation of an invariant for TD systems increases with respect to both its order and the dimensionality of the system, here we present the derivation only of third-order invariants by using the rationalization method in terms of the complex coordinates. The corresponding results for the Cartesian case can be derived in an analogous manner.

We consider the system described by the Lagrangian (74) and now make an ansatz for the third-order invariant [cf. equation (56)] as

$$I = a_0 + a_i\xi_i + (1/2)a_{ij}\xi_i\xi_j + (1/6)a_{ijk}\xi_i\xi_j\xi_k \tag{116}$$

where $\xi_i = \dot{Z}$ and $\xi_2 = \dot{\bar{Z}}$, and a_0, a_i, a_{ij}, a_{ijk} are functions of Z, $\bar{Z}$, and t as before. For the invariance of I, one uses (5) and (6) and obtains an identity. Now equating the coefficients of the powers of ξ_1 and ξ_2 and their products to zero after accounting for the proper symmetrization in the resultant expression, one obtains the following relations for the a's as before:

$$a_{ijk,l} + a_{jkl,i} + a_{kli,j} + a_{lij,k} = 0 \tag{117}$$

$$a_{ij,k} + a_{jk,i} + a_{ki,j} + (\partial a_{ijk}/\partial t) = 0 \tag{118}$$

$$a_{i,j} + a_{j,i} + (\partial a_{ij}/\partial t) + a_{ijk}\dot{\xi}_k = 0 \tag{119}$$

$$a_{0,i} + \partial a_i/\partial t + a_{ij}\dot{\xi}_j = 0 \tag{120}$$

$$(\partial a_0/\partial t) + a_i\dot{\xi}_i = 0 \tag{121}$$

Now, using $\dot{\xi}_1 = -2(\partial V/\partial \bar{Z})$, $\dot{\xi}_2 = -2(\partial V/\partial Z)$, we obtain that (117)–(121) yield the following set of coupled PDEs:

$$(\partial a_{111}/\partial Z) = 0 \tag{122a}$$

$$(\partial a_{222}/\partial \bar{Z}) = 0 \tag{122b}$$

$$(\partial a_{111}/\partial \bar{Z}) + 3(\partial a_{112}/\partial Z) = 0 \tag{122c}$$

$$(\partial a_{112}/\partial \bar{Z}) + (\partial a_{122}/\partial Z) = 0 \tag{122d}$$

$$(\partial a_{222}/\partial Z) + 3(\partial a_{122}/\partial \bar{Z}) = 0 \tag{122e}$$

$$3(\partial a_{11}/\partial Z) + (\partial a_{111}/\partial t) = 0 \tag{122f}$$

$$(\partial a_{11}/\partial \bar{Z}) + 2(\partial a_{12}/\partial Z) + (\partial a_{112}/\partial t) = 0 \tag{122g}$$

$$(\partial a_{22}/\partial Z) + 2(\partial a_{12}/\partial \bar{Z}) + (\partial a_{122}/\partial t) = 0 \tag{122h}$$

$$3(\partial a_{22}/\partial \bar{Z}) + (\partial a_{222}/\partial t) = 0 \tag{122i}$$

$$2(\partial a_1/\partial Z) + (\partial a_{11}/\partial t) = 2a_{111}(\partial V/\partial \bar{Z}) + 2a_{112}(\partial V/\partial Z) \quad (122j)$$

$$2(\partial a_2/\partial \bar{Z}) + (\partial a_{22}/\partial t) = 2a_{122}(\partial V/\partial \bar{Z}) + 2a_{222}(\partial V/\partial Z) \quad (122k)$$

$$(\partial a_1/\partial \bar{Z}) + (\partial a_2/\partial Z) + (\partial a_{12}/\partial t) = 2a_{112}(\partial V/\partial \bar{Z}) + 2a_{122}(\partial V/\partial Z) \quad (122l)$$

$$(\partial a_0/\partial Z) + (\partial a_1/\partial t) = 2a_{11}(\partial V/\partial \bar{Z}) + 2a_{12}(\partial V/\partial Z) \quad (122m)$$

$$(\partial a_0/\partial \bar{Z}) + (\partial a_2/\partial t) = 2a_{12}(\partial V/\partial \bar{Z}) + 2a_{22}(\partial V/\partial Z) \quad (122n)$$

$$(\partial a_0/\partial t) = 2a_1(\partial V/\partial \bar{Z}) + 2a_2(\partial V/\partial Z) \quad (122o)$$

Note that for $a_{ijk} = 0$ in (116), these 15 equations reduce to 10 equations analyzed earlier (Sect. 4.1.2) for the second-order case. Now, we present the solutions of these equations for determining the coefficient functions a's.

5.2.1. *Determination of a_{ijk}*

Clearly, (122a) and (122b) imply that $a_{111} \equiv a_{111}(\bar{Z}, t) = \bar{\Psi}_1(\bar{Z}, t)$ and $a_{222} = a_{222}(Z, t) = \Phi_1(Z, t)$. On differentiating (122c) w.r.t. $\bar{Z}$ and using (122d), one obtains

$$(\partial^2 a_{111}/\partial \bar{Z}^2) - 3(\partial^2 a_{122}/\partial Z^2) = 0$$

and similarly, the differentiation of (122e) w.r.t. Z yields

$$(\partial^2 a_{222}/\partial Z^2) + 3(\partial^2 a_{122}/\partial Z \cdot \partial \bar{Z}) = 0$$

Further differentiation of these two equations w.r.t. $\bar{Z}$ and Z, respectively, after noting that $(\partial^3 a_{122}/\partial \bar{Z} \cdot \partial Z^2) = (\partial^3 a_{122}/\partial Z^2 \cdot \partial \bar{Z})$, leads to

$$\begin{aligned}(\partial^3 a_{111}/\partial \bar{Z}^3) &= -(\partial^3 a_{222}/\partial Z^3)\\ &= \text{a function of } t \text{ alone} \quad [\text{say } \sigma_1(t)]\end{aligned} \quad (123)$$

If we assume the separability of Ψ_1 and Φ_1 as

$$\Psi_1(\bar{Z}, t) = \psi_1(\bar{Z})f_1(t); \qquad \Phi_1(Z, t) = \phi_1(Z)g_1(t)$$

then the solution to (123) can be obtained immediately as

$$a_{111} = (1/6)\sigma_1\bar{Z}^3 + (1/2)\sigma_2\bar{Z}^2 + \sigma_3\bar{Z} + \sigma_4 \quad (124a)$$

$$a_{222} = -(1/6)\sigma_1 Z^3 + (1/2)\sigma_5 Z^2 + \sigma_6 Z + \sigma_7 \quad (124b)$$

The coefficient functions a_{112} and a_{122} can be obtained in the same way from the integrations of (122c) and (122e) as

$$a_{112} = -\tfrac{1}{6}\sigma_1\bar{Z}^2 Z - \tfrac{1}{3}\sigma_2\bar{Z}Z + \tfrac{1}{6}\sigma_5\bar{Z}^2 - \tfrac{1}{3}\sigma_3 Z + \sigma_8\bar{Z} + \sigma_9 \quad (124c)$$

$$a_{122} = \tfrac{1}{6}\sigma_1 Z^2\bar{Z} - \tfrac{1}{3}\sigma_5\bar{Z}Z + \tfrac{1}{6}\sigma_2 Z^2 - \tfrac{1}{3}\sigma_6\bar{Z} + \sigma_8 Z + \sigma_{10} \quad (124d)$$

Regarding the notations, note that throughout this subsection the arbitrary functions Ψ_i are functions of $\bar{Z}$ and t, and Φ_i are functions of Z and t; ψ_i are functions of $\bar{Z}$ alone, ϕ_i are functions of Z alone, and σ_i are functions of t alone. Further, σ_1 is the separation function and other σ_i are arbitrary functions of integration.

5.2.2. *Determination of* a_{ij}

To obtain a_{ij} we use (124a) and (124b) in (122f) and (122i), respectively, and by integrating the resultant equations we obtain the expressions for a_{11} and a_{22} as

$$a_{11} = -(1/18)\,\dot{\sigma}_1 Z\bar{Z}^3 - \tfrac{1}{6}\dot{\sigma}_2 Z\bar{Z}^2 - \tfrac{1}{3}\dot{\sigma}_3 Z\bar{Z} - \tfrac{1}{3}\dot{\sigma}_4 Z + \Psi_3(\bar{Z}, t) \quad (125a)$$

$$a_{22} = (1/18)\dot{\sigma}_1 Z^3\bar{Z} - \tfrac{1}{6}\dot{\sigma}_5 Z^2\bar{Z} - \tfrac{1}{3}\dot{\sigma}_6 Z\bar{Z} - \tfrac{1}{3}\dot{\sigma}_7\bar{Z} + \Phi_3(Z, t) \quad (125b)$$

These expressions, when used respectively in (122g) and (122h) along with (124c) and (124d), yield the two expressions for a_{12} involving some arbitrary functions, Ψ_3, Ψ_4 and Φ_3, Φ_4. A comparison of these two expressions for a_{12} provides the identity

$$\begin{aligned}
&\tfrac{1}{6}\dot{\sigma}_1 Z^2\bar{Z}^2 - \tfrac{1}{4}\dot{\sigma}_5\bar{Z}^2 Z + \tfrac{1}{4}\dot{\sigma}_2 Z^2\bar{Z} + \tfrac{1}{6}\dot{\sigma}_3 Z^2 - \tfrac{1}{6}\dot{\sigma}_6\bar{Z}^2 \\
&- \dot{\sigma}_8 Z\bar{Z} - \tfrac{1}{2}\dot{\sigma}_9 Z + \tfrac{1}{2}\dot{\sigma}_{10}\bar{Z} - \tfrac{1}{2}Z(\partial\Psi_3/\partial\bar{Z}) \\
&+ \Psi_4 + \tfrac{1}{2}\bar{Z}(\partial\Phi_3/\partial Z) + \Phi_4 = 0
\end{aligned} \quad (126)$$

From this identity one immediately obtains

$$\dot{\sigma}_1 = 0 \qquad \text{or} \qquad \sigma_1 = \text{const} \qquad (\text{say } c_1) \quad (127a)$$

and subsequently

$$(\partial^3\Psi_3/\partial\bar{Z}^3) = -(1/2)\dot{\sigma}_5; \qquad (\partial^3\Phi_3/\partial Z^3) = -(1/2)\dot{\sigma}_2$$

which imply

$$\Psi_3(\bar{Z}, t) = (-1/12)\dot{\sigma}_5\,\bar{Z}^3 + (1/2)\sigma_{11}\bar{Z}^2 + \sigma_{12}\bar{Z} + \sigma_{13}$$

$$\Phi_3(Z, t) = (-1/12)\dot{\sigma}_2\,Z^3 + (1/2)\sigma_{14}Z^2 + \sigma_{15}Z + \sigma_{16}$$

from which Ψ_4 and Φ_4 in (126) can be set as

$$\Psi_4(\bar{Z}, t) = \tfrac{1}{6}\dot{\sigma}_6\bar{Z}^2 - \tfrac{1}{2}\dot{\sigma}_{10}\bar{Z}; \qquad \Phi_4(Z, t) = \tfrac{1}{6}\dot{\sigma}_3 Z^2 - \tfrac{1}{2}\dot{\sigma}_9 Z$$

With these results, the two expressions for a_{12} become

$$a_{12} = (1/6)\dot{\sigma}_2\bar{Z}Z^2 + (1/24)\dot{\sigma}_5\bar{Z}^2Z + (1/6)\dot{\sigma}_3Z^2 + (1/6)\dot{\sigma}_6\bar{Z}^2 - (1/2)\,(\dot{\sigma}_8 + \sigma_{11})Z\bar{Z} - (1/2)(\dot{\sigma}_9 + \sigma_{12})Z - (1/2)\dot{\sigma}_{10}\bar{Z}$$

and

$$a_{12} = (1/24)\dot{\sigma}_2\,\bar{Z}Z^2 + (1/6)\dot{\sigma}_5Z\bar{Z}^2 + (1/6)\dot{\sigma}_3Z^2 + (1/6)\dot{\sigma}_6\bar{Z}^2 + (1/2)(\dot{\sigma}_8 - \sigma_{14})Z\bar{Z} - (1/2)(\dot{\sigma}_{10} + \sigma_{15})\bar{Z} - (1/2)\dot{\sigma}_9Z$$

Uniqueness of these two expressions will further require that

$$\dot{\sigma}_2 = \dot{\sigma}_5 = 0 \tag{127b}$$

$$\sigma_{12} = \sigma_{15} = 0 \tag{127c}$$

$$2\dot{\sigma}_8 + \sigma_{11} - \sigma_{14} = 0 \tag{127d}$$

These equations imply σ_2 = const (say c_2) and σ_5 = const (say c_5). Finally, the coefficients a_{11}, a_{22}, and a_{12} become

$$a_{11} = -\tfrac{1}{3}\,\dot{\sigma}_3Z\bar{Z} - \tfrac{1}{3}\dot{\sigma}_4Z + \tfrac{1}{2}\sigma_{11}\bar{Z}^2 + \sigma_{13} \tag{128a}$$

$$a_{22} = -\tfrac{1}{3}\dot{\sigma}_6Z\bar{Z} - \tfrac{1}{3}\dot{\sigma}_7\bar{Z} + \tfrac{1}{2}\sigma_{14}Z^2 + \sigma_{10} \tag{128b}$$

$$a_{12} = \tfrac{1}{6}\dot{\sigma}_3Z^2 + \tfrac{1}{6}\dot{\sigma}_6\bar{Z}^2 - \tfrac{1}{2}\dot{\sigma}_9Z - \tfrac{1}{2}\dot{\sigma}_{10}\bar{Z} \tag{128c}$$

and the expressions for a_{ijk} from (124) take the form

$$a_{111} = \tfrac{1}{6}c_1\bar{Z}^3 + \tfrac{1}{2}c_2\bar{Z}^2 + \sigma_3\bar{Z} + \sigma_4 \tag{129a}$$

$$a_{222} = -\tfrac{1}{6}c_1Z^3 + \tfrac{1}{2}c_5Z^2 + \sigma_6Z + \sigma_7 \tag{129b}$$

$$a_{112} = -\tfrac{1}{6}c_1\bar{Z}^2Z - \tfrac{1}{3}c_2\bar{Z}Z + \tfrac{1}{6}c_5\bar{Z}^2 - \tfrac{1}{3}\sigma_3Z + \sigma_8\bar{Z} + \sigma_9 \tag{129c}$$

$$a_{122} = \tfrac{1}{6}c_1Z^2\bar{Z} - \tfrac{1}{3}c_5Z\bar{Z} + \tfrac{1}{6}c_2Z^2 - \tfrac{1}{3}\sigma_6\bar{Z} - \sigma_8Z + \sigma_{10} \tag{129d}$$

5.2.3. Derivation of the "Potential" Equations

For the terms involving the potential in (122), we introduce the following notations for convenience:

$$F = a_{112}(\partial V/\partial\bar{Z}) + a_{122}(\partial V/\partial Z)$$

$$G_1 = a_{122}(\partial V/\partial\bar{Z}) + a_{222}(\partial V/\partial Z)$$

$$G_2 = a_{111}(\partial V/\partial\bar{Z}) + a_{112}(\partial V/\partial Z)$$

$$R = a_{11}(\partial V/\partial \bar{Z}) + a_{12}(\partial V/\partial Z)$$
$$S = a_{12}(\partial V/\partial \bar{Z}) + a_{22}(\partial V/\partial Z) \qquad (130)$$

Differentiating (122l) w.r.t. $\bar{Z}$ and using (122k) for $(\partial a_2/\partial \bar{Z})$, one obtains

$$(\partial^2 a_1/\partial \bar{Z}^2) = -\partial[(\partial a_{12}/\partial \bar{Z}) - \tfrac{1}{2}(\partial a_{22}/\partial Z)]/\partial t + 2(\partial F/\partial \bar{Z}) - (\partial G_1/\partial Z)$$

which, after using (128b) and (128c), reduces to

$$(\partial^2 a_1/\partial \bar{Z}^2) = -\tfrac{1}{2}(\ddot{\sigma}_6 \bar{Z} - \dot{\sigma}_{14} Z - \ddot{\sigma}_{10}) + 2(\partial F/\partial Z) - (\partial G_1/\partial Z) \qquad (131)$$

Similarly (122l), after using (122j), (128a), and (128c), becomes

$$(\partial^2 a_2/\partial Z^2) = -\tfrac{1}{2}(\ddot{\sigma}_3 Z - \ddot{\sigma}_{11}\bar{Z} - \ddot{\sigma}_9) + 2(\partial F/\partial Z) - (\partial G_2/\partial \bar{Z}) \quad (132)$$

Thus, the coefficients a_1 and a_2 can be computed by integrating (131) and (132), respectively.

On differentiating (122m) and (122n) w.r.t. $\bar{Z}$ and Z, respectively, and then using $(\partial^2 a_0/\partial Z \cdot \partial \bar{Z}) = (\partial^2 a_0/\partial \bar{Z} \cdot \partial Z)$ for eliminating a_0, one obtains

$$\frac{\partial}{\partial t}(\partial a_1/\partial \bar{Z}) - \frac{\partial}{\partial t}(\partial a_2/\partial Z) = 2[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] \qquad (133a)$$

Similarly, differentiation of (122l) w.r.t. t gives

$$\frac{\partial}{\partial t}(\partial a_1/\partial \bar{Z}) + \frac{\partial}{\partial t}(\partial a_z/\partial Z) = -(\partial^2 a_{12}/\partial t^2) + 2(\partial F/\partial t) \qquad (133b)$$

Now, (133a) and (133b) give rise to

$$\frac{\partial}{\partial t}(\partial a_1/\partial \bar{Z}) = -(1/2)(\partial^2 a_{12}/\partial t^2) + (\partial \mathrm{F}/\partial t) + [(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] \; (134a)$$

$$\frac{\partial}{\partial t}(\partial a_2/\partial Z) = -(1/2)(\partial^2 a_{12}/\partial t^2) + (\partial \mathrm{F}/\partial t) + [(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] \; (134b)$$

In order to eliminate a_1 and a_2 from (134a) and (134b), we differentiate them w.r.t. Z and $\bar{Z}$, respectively, and then correspondingly use the results (131) and (132). This gives [50] the following pair of "potential" equations:

$$(\partial^2 F/\partial t \cdot \partial \bar{Z}) - (\partial^2 G_1/\partial t \cdot \partial Z) - (1/2)(\ddot{\sigma}_6 \bar{Z} - \ddot{\sigma}_{14} Z - \ddot{\sigma}_{10}$$
$$= (\partial/\partial \bar{Z})[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] - (1/2)(\partial^3 a_{12}/\partial \bar{Z} \cdot \partial t^2) \qquad (135)$$

and

$$(\partial^2 F/\partial t \cdot \partial Z) - (\partial^2 G_2/\partial t \cdot \partial \bar{Z}) - (1/2)(\ddot{\sigma}_3 Z - \ddot{\sigma}_{11}\bar{Z} - \ddot{\sigma}_9)$$
$$= -(\partial/\partial Z)[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] - (1/2)(\partial^3 a_{12}/\partial Z \cdot \partial t^2) \quad (136)$$

Alternative and somewhat simple-looking forms of these equations can be obtained by differentiating (135) and (136) once again w.r.t. Z and $\bar{Z}$, respectively, and then noting that

$$\frac{\partial^2}{\partial Z \cdot \partial \bar{Z}}[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)] = \frac{\partial^2}{\partial \bar{Z} \cdot \partial Z}[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)]$$

and

$$(\partial^4 a_{12}/\partial \bar{Z} \cdot \partial \bar{Z} \cdot \partial t^2) = (\partial^4 a_{12}/\partial \bar{Z} \cdot \partial Z \cdot \partial t^2) = 0$$

[cf. equation (128c)]. Equations (135) and (136) reduce to

$$\frac{\partial}{\partial t}[(\partial^2 G_1/\partial Z^2) + (\partial^2 G_2/\partial \bar{Z}^2)] - 2(\partial^3 F/\partial t \cdot \partial Z \cdot \partial \bar{Z})$$
$$= (1/2)(\ddot{\sigma}_{11} + \ddot{\sigma}_{14}) \quad (135')$$

$$\frac{\partial}{\partial t}[(\partial^2 G_1/\partial Z^2) - (\partial^2 G_2/\partial \bar{Z}^2)] + 2\frac{\partial^2}{\partial Z \cdot \partial \bar{Z}}[(\partial R/\partial \bar{Z}) - (\partial S/\partial Z)]$$
$$= -(1/2)(\ddot{\sigma}_{11} + \ddot{\sigma}_{14}) \quad (136')$$

Recall that F, G_1, G_2, R, and S involve derivatives of the potential function [cf. (130)] V. As a result, the solutions of (135) and (136) [or of (135′) and (136′)] would directly provide systems admitting the third-order first invariant. Further, these solutions have to be in conformity with (122m), (122n), and (122o). This, in fact, will help in fixing the other arbitrary functions as before.

5.2.4. Results in the Cartesian Case

It is not difficult to derive the "potential" equations similar to (135′) and (136′) in Cartesian coordinates. The results are practically the same as (135′) and (136′) with the coefficient functions same as defined in (128) and (129), but now Z and $\bar{Z}$ are replaced by x_2 and x_1 coordinates, respectively. For this purpose the other changes to be noted are (i) the replacement of $\partial R/\partial \bar{Z} - \partial S/\partial Z$ in (136′) by $\partial R/\partial x_1 - \partial S/\partial x_2$, and (ii) the revised definitions of F, G_1, G_2, R, and S as

$$F = a_{112}(\partial V/\partial x_1) + a_{122}(\partial V/\partial x_2)$$

$$G_1 = a_{122}(\partial V/\partial x_1) + a_{222}(\partial V/\partial x_2)$$

$$G_2 = a_{111}(\partial V/\partial x_1) + a_{112}(\partial V/\partial x_2)$$

$$R = a_{11}(\partial V/\partial x_1) + a_{12}(\partial V/\partial x_2)$$

$$S = a_{22}(\partial V/\partial x_2) + a_{12}(\partial V/\partial x_1)$$

6. ERMAKOV SYSTEMS

6.1. Generalized Ermakov Systems

In Section 3.2, the applications of the Ermakov method, employed initially only for the simple TDHO problem, to more generalized systems were already pointed out in terms of (30). As a special case, in this connection, while $x(t)$ satisfies (30a), $\rho(t)$ is found[28] to satisfy

$$\ddot{\rho} + \omega^2(t)\rho = (1/x\rho^2) \sum_i c_i (x/\rho)^{2m_i - 1}, \qquad (i = 1, 2) \tag{137}$$

where c_i and m_i are arbitrary constants. For $i = 1$ and 2, and with $m_1 = (m - 2)/2$, $m_2 = (2m - 2)/2$, equation (137) becomes

$$\ddot{\rho} + \omega^2(t)\rho = c_1 x^{m-4} \rho^{1-m} + c_2{}^{2m-4} \rho^{1-2m} \tag{138}$$

Among other generalizations of Ermakov systems considered by Reid and Ray,[32] one is in relation with the nonlinear superposition law for the solutions of higher order nonlinear equations. We avoid the discussion of such generalizations here. It may be mentioned that all the above generalizations of Ermakov systems are essentially for 1D TD systems, particularly for the 1D TDHO, and $\rho(t)$ appears as an auxiliary variable needed to provide the invariant for the corresponding TD system in (1 + 1) dimensions. On the other hand, $\rho(t)$ also plays a specific role while looking for a physical interpretation of the derived invariant (cf. Section 8). In any case, it does not imply the generalization of Ermakov systems to higher space dimension in the present context of (2 + 1) dimensions.

Other generalizations of Ermakov systems have recently been considered by Leach[51] and Athorne.[52] Leach[51] finds an explanation for the nature of the Ermakov system described by

$$\ddot{x}_1 + \omega^2(t)x_1 = g_1(x_2/x_1)/x_1^3 \tag{139a}$$

$$\ddot{x}_2 + \omega^2(t)x_2 = f_1(x_2/x_1)/x_2^3 \tag{139b}$$

in terms of the symmetry algebra $sl(2, \mathcal{R})$. In this case, a transformation of time and space variables, namely $T = \cot(\int \rho^{-2}\, dt)$, $X = \rho^{-1} x_1 \csc T$, $Y = \rho^{-1} x_2 \csc T$, eliminates $\omega^2(t)$ from (139), and the newly introduced variable ρ is found to satisfy an equation of the type (15) with $k = 1$. The generalized Ermakov systems are really Cartesian forms of a system of equations, but for their deeper understanding from symmetry considerations, the corresponding polar forms[51]

$$\ddot{r} - r\dot{\theta}^2 = F(\theta)/r^3; \qquad r\ddot{\theta} + 2\dot{r}\,\dot{\theta} = G(\theta)/r^3 \tag{140}$$

turn out to be more convenient. In fact, the symmetries explored, in this case

$$G_1 = \partial/\partial t; \qquad G_2 = 2t(\partial/\partial t) + r(\partial/\partial r); \qquad G_3 = t^2(\partial/\partial t) + tr(\partial/\partial r) \tag{141}$$

correspond, respectively, to time-translation, self-similar, and conformal transformations. Interestingly, if the system has Hamiltonian structure [the necessary condition for this is that $G = -\frac{1}{2}(\partial F/\partial\theta)$ in (140)], then the angular component of the equations of motion (140) directly gives rise to the Ermakov invariant, i.e.,

$$I = \tfrac{1}{2}(r^2\dot{\theta})^2 - \int G(\theta)\, d\theta = \tfrac{1}{2}[p_\theta^2 + F(\theta)] \tag{142}$$

Not only this, the polar forms also suggest an easy way to generalize the Ermakov systems to higher dimensions. For example, in the 3D case, one can write[51] the equations of motion possessing the $sl(2, \mathcal{R})$ symmetry as

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 &= F(\theta, \phi)/r^3 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta &= G(\theta, \phi)/r^3 \\ r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta &= R(\theta, \phi)/r^3 \sin\theta \end{aligned} \tag{143}$$

The system has a Hamiltonian provided

$$G(\theta, \phi) = -(1/2)(\partial F/\partial\theta), \qquad R(\theta, \phi) = -(1/2)(\partial F/\partial\phi)$$

This leads to the potential term as $V = F(\theta, \phi)/2r^2$. The first invariant for this system was derived by Leach.[51] For further studies of 3D systems we refer to the recent work of Govinder and Leach.[53]

Athorne[52] makes use of the symmetry algebra of Leach and analyzes a Kepler–Ermakov system of the type $\ddot{\underline{x}} + \omega^2(t)\underline{x} = v(\underline{x})r^3$ by setting $\omega^2(t) = 0$. Another interesting system studied in this context is that of coupled Pinney equations[54]

$$\ddot{x}_1 + \omega^2(t)x_1 = \beta x_1^{-3} - \alpha x_1 x_2^{-4} \tag{144a}$$

$$\ddot{x}_2 + \omega^2(t)x_2 = \delta x_2^{-3} - \gamma x_2 x_1^{-4} \tag{144b}$$

In this case, however, there remains,[52] in general, a difficulty of finding a Hamiltonian structure. In the absence of such a structure the system does not warrant much physical interest. For example, for the system (144) the Hamiltonian structure exists only for $\alpha = 3\delta$, $\gamma = 3\beta$ and with a kinetic term of hardly any physical interest. In spite of the fact that the invariant associated with this system can be constructed explicitly, this system does not turn out[52] to be integrable even for $\omega(t) = \text{const}$, i.e., even in a TID case. Depending upon the character of their superposition laws [i.e., on the nature of the functions f_1 and g_1 in (139)] such as rational, algebraic, or automorphic, etc., the Ermakov systems can also be classified. For such details, we refer to the work of Athorne[55] and Govinder *et al.*[56]

It is now clear that the Ermakov systems could be of both types—those which have Hamiltonian structure and those which do not have such a structure. From the recent group-theoretic studies of generalized coupled systems by Govinder and Leach,[53] it appears that there can further be two separate situations—one in which the system admits an invariant of angular momentum type (i.e., Ermakov type[30]) and another in which the invariant is of energy integral type (i.e., Lewis type[18]). However, we do not treat such details here.

6.2. New Ermakov-Type Systems

The variable $x_2(t)$ in (144) can, in principle, be regarded as $\rho(t)$ of (15) or of (138) and is in fact a variable in the second space dimension. On the other hand, the system studied by Leach[51] [cf. (139)] reduces to the Ray and Reid form (30) on redefining g_1 and f_1 as

$$g_1(x_2/x_1) = (x_1/x_2)g(x_1/x_2)$$

$$f_1(x_2/x_1) = (x_2/x_1)f(x_2/x_1)$$

but becomes structurally different from (3.20) in the sense that the ρ equation now appears as a by-product of the transformation. Thus, (139) and (144) describe certain TD coupled oscillators in 2D (with equal spring constants along each dimension), but without any auxiliary equation. It may be of interest to look for Ermakov or Ermakov-type systems which remain integrable and also retain the Hamiltonian structure. In fact, the conditions under which the Ermakov system (30) is also a Hamiltonian system are investigated by Cervero and Lejarreta,[57] but by treating the auxiliary variable ρ on a par with the space variable x. Recently, inspired by the results of the dynamical algebraic approach (cf. Section 4.2), the author has proposed[58] a class of

Ermakov-type systems in 2D which (i) deals with unequal but related spring constants, (ii) ensures in general a Hamiltonian structure, (iii) admits second-order invariants, (iv) can involve fractional powers in the coupling terms, and (v) involves a pair of auxiliary equations in a natural manner. All these features could be important in characterizing new Ermakov-type systems which in fact are capable of describing[58] TD anharmonic and anisotropic oscillators in 2D.

Here we just draw attention to the systems (102) and (103), which are special cases of the system described by the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \alpha_1(t)x_1^2 - \alpha_2(t)x_2^2 - \beta_{10}x_1^{-2-n}x_2^n - \beta_{20}x_1^n x_2^{-2-n} \tag{145}$$

with the corresponding pair of coupled equations of motion

$$\begin{aligned} \ddot{x}_1 + 2\alpha_1(t)x_1 &= (n+2)\beta_{10}x_1^{-3-n}x_2^n - n\beta_{20}x_1^{n-1}x_2^{-2-n} \\ \ddot{x}_2 + 2\alpha_2(t)x_2 &= -n\beta_{10}x_1^{-2-n}x_2^{n-1} + (n+2)\beta_{20}x_1^n\, x_2^{-3-n} \end{aligned} \tag{146$'$}$$

The first invariant for this system is given by[58]

$$\begin{aligned} I = {} & \tfrac{1}{2}\psi(p_1^2 + p_2^2) + (\tfrac{1}{4}\ddot{\psi} + \alpha_1\psi)x_1^2 \\ & + (\tfrac{1}{4}\ddot{\psi} + \alpha_2\psi)x_2^2 + \beta_{10}\psi x_1^{-2-n}\, x_2^{n-1} + \beta_{20}\psi x_1^n\, x_2^{-2-n} \\ & - \tfrac{1}{2}\dot{\psi}(x_1 p_1 + x_2 p_2) \end{aligned} \tag{147}$$

where n is an arbitrary number; ψ is given as [cf. (97)] $\psi = c(\alpha_1 - \alpha_2)^{-1/2}$, and α_1 and α_2 are related through (73), as before.

7. INTEGRABILITY OF NONCENTRAL TD SYSTEMS IN TWO DIMENSIONS

In the preceding sections we have constructed only one invariant for both 1D and 2D TD systems. By constructing one invariant for 1D TD systems, their integrability according to Whittaker's conjecture, is established, but this is not so for the 2D TD systems. Although some efforts have been made[59] in this context, the TID systems studied are those having some kind of radial symmetry. In this latter case, particularly for the NC systems, there remains a problem of obtaining the second invariant I_2, which should not only be independent of the first one I_1 (in the sense of the involuting property), but should also conform to the conditions (5) and (6) for the given Hamiltonian H of the system. As a matter of fact one should have

$$[I_1, H]_{\mathrm{PB}} + (\partial I_1/\partial t) = 0 \tag{148a}$$

$$[I_2, H]_{\mathrm{PB}} + (\partial I_2/\partial t) = 0 \tag{148b}$$

$$[I_1, I_2]_{\text{PB}} = 0 \tag{148c}$$

where H is now given by

$$H = (1/2)(p_1^2 + p_2^2) + V(x_1, x_2, t) \tag{149}$$

For this purpose, all the methods discussed in Section 4 in terms of the simple polynomial ansatz start exhibiting complexity from the point of view of constructing higher order invariants, but somehow do not even ensure the existence of I_2. In view of this complexity, although it becomes difficult to question the viability of these methods as far as the construction of I_2 is concerned, the fact is that there does not exist a general criterion to guarantee the integrability of the higher dimensional TD systems, even in terms of nonpolynomial form of the invariant. In what follows, we make an attempt to set some further restrictions on the coefficient functions in the polynomial ansatz by assuming the existence of the second invariant I_2 for TD systems in 2D.

It is true that the expressions obtained for the coefficient functions in general involve several arbitrary functions (of time)/constants which are normally set in accordance with the parameters of the potential function V. Very often, these arbitrary functions/constants outnumber the parameters of the system and a lot of freedom is left to fix at least a few of them. Therefore, the spirit in the following derivation is to impose some further restrictions on the coefficient functions in view of (148) along with the ones already required for the derivation of a particular invariant using the rationalization method. The results are given only for a few cases differing mainly due to the different orders of I_1 and I_2; in particular, we consider the cases (L, L), (L, Q), (L, C), (Q, Q), (Q, C), and (C, C), where L, Q, and C, respectively, stand for the linear, quadratic, or cubic nature of (I_1, I_2). We first give results for the (C, C) case and subsequently obtain the results for other cases by setting some of the coefficient functions to zero.

(i)(C, C) case: For the Hamiltonian (149) and the ansatze

$$I_1 = a_0 + a_i\xi_i + (1/2)a_{ij}\xi_i\xi_j + (1/6)a_{ijk}\xi_i\xi_j\xi_k \tag{150}$$

$$I_2 = b_0 + b_i\xi_i + (1/2)b_{ij}\xi_i\xi_j + (1/6)b_{ijk}\xi_i\xi_j\xi_k \tag{151}$$

while the requirements (148a) and (148b) lead to the same set of PDEs for both functions a and b as (117)–(121), the use of (148c), in the spirit of (8), gives rise to the restrictions on them as

$$a_{ijk,m}b_{lnm} - b_{ijk,m}a_{lnm} = 0 \tag{152}$$

$$a_{ijk,m}b_{lm} + 3a_{ij,m}b_{klm} - b_{ijk,m}a_{lm} - 3b_{ij,m}a_{klm} = 0 \tag{153}$$

$$a_{ijk,m}b_m + 3a_{ij,m}b_{km} + 6a_{i,m}b_{kjm} - b_{ijk,m}a_m - 3b_{ij,m}a_{km} - 6b_{i,m}a_{kjm} = 0 \quad (154)$$

$$a_{ij,m}b_m + 2a_{i,m}b_{jm} + 2a_{0,m}b_{ijm} - b_{ij,m}a_m - 2b_{i,m}a_{jm} - 2b_{0,m}a_{ijm} = 0 \quad (155)$$

$$a_{i,m}b_m + a_{0,m}b_{im} - b_{i,m}a_m - b_{0,m}a_{im} = 0 \quad (156)$$

$$a_{0,m}b_m - b_{0,m}a_m = 0 \quad (157)$$

In what follows we give similar results [mainly as special cases of (152)–(157)] for situations when the pair of involuting invariants are of different (but less than three) orders in momenta.

(ii) (Q, C) case: By setting $a_{ijk} = 0$ throughout (150)–(157) one can obtain the conditions on a's and b's. In fact, in this case, while (152) does not appear and (156) and (157) remain the same, (153)–(155), respectively, take the form

$$3a_{ij,m}b_{klm} - b_{ijk,m}a_{lm} = 0$$

$$3a_{ij,m}b_{km} + 6a_{i,m}b_{kjm} - b_{ijk,m}a_m - 3b_{ij,m}a_{km} = 0$$

$$a_{ij,m}b_m + 2a_{i,m}b_{jm} + 2a_{0,m}b_{ijm} - b_{ij,m}a_m - 2b_{i,m}a_{jm} = 0$$

(iii) (L, C) case: Set a_{ijk} and a_{ij} to zero in (150)–(157). As a result, (152) and (153) are absent in this case and (157) remains unchanged, leaving (154)–(156), respectively, in the form

$$6a_{i,m}b_{kjm} - b_{ijk,m}a_m = 0$$

$$2a_{i,m}b_{jm} + 2a_{0,m}b_{ijm} - a_{ij,m}a_m = 0$$

$$a_{i,m}b_m + a_{0,m}b_{im} - b_{i,m}a_m = 0$$

(iv) (Q, Q) case: Set $a_{ijk} = b_{ijk} = 0$ in (150)–(157). Consequently, (152) and (153) now do not appear and (156) and (157) remain the same. Equations (154) and (155) in this case take the form, respectively,

$$a_{ij,m}b_{km} - b_{ij,m}a_{km} = 0$$

$$a_{ij,m}b_m + 2a_{i,m}b_{jm} - b_{ij,m}a_m - 2b_{i,m}a_{jm} = 0$$

(v) (L, Q) case: Set $a_{ijk} = a_{ij} = b_{ijk} = 0$ in (150)–(157). In this case, (152)–(154) are absent and (157) remains unaffected. Equations (155) and (156). respectively, now reduce to the forms

$$2a_{i,m}b_{jm} - b_{ij,m}a_m = 0$$

$$a_{i,m}b_m + a_{0,m}b_{im} - b_{i,m}a_m = 0$$

(vi) (L, L) case: For this simplest case, one sets $a_{ijk} = a_{ij} = b_{ijk} = b_{ij} = 0$ in (150)–(157). as before, and obtains (156) in the form $a_{i,m}b_m - b_{i,m}a_m = 0$ in addition to (157). In this case (152)–(155) are absent.

Thus, some of the arbitrary functions/constants left undetermined in the expressions for the a's in Section 5.2 (and now in the similar expressions for the b's) for the third- (or lower) order case can be fixed with the help of the above restrictions on the a's and b's. This suggests a possibility of getting an involutively independent second invariant I_2 for the system (149) and thereby implies the integrability of this system.

8. THE ROLE AND SCOPE OF DYNAMICAL INVARIANTS IN PHYSICAL PROBLEMS: INTERPRETATION AND APPLICATIONS

The invariants when defined in a broader sense play an important role in different branches of mathematics and mathematical physics, but somehow the description of physical reality limits their applications in physics and other allied sciences. In this respect while the role and the scope of the dynamical invariants for the 1D systems have been discussed at great length in the literature, they have not been explored to that extent for two- and higher dimensional systems. Here, we briefly highlight some possible physical interpretations known for the invariant for the TD HO system in 1D and also discuss some physical situations in different branches of mathematical sciences where not only does the role of these invariants become transparent, but also their study may find some applications.

8.1. Physical Interpretations of Dynamical Invariants

From the point of view of having an in-depth study of a dynamical system, no doubt it is desirable to know all of its permissible invariants if they exist, but as far as the assigning of physical meaning to these invariants is concerned, it has not been possible even for all the available ones. Further, the assignment of such a physical meaning to a complicated mathematical form of an invariant appears to be difficult, in general, but for the polynomial form such possibilities have been explored. In particular, we list here some plausible interpretations suggested for the form (14), which corresponds to a TD HO system.

(i) According to Eliezer and Gray,[61] the constancy of I is equivalent to the constancy of the angular momentum associated with the auxiliary equation (15). The motion in a straight line described by $\ddot{x} + \omega^2(t)x = 0$ can be viewed as the projection of the 2D motion of a particle under an attractive center of force. Then the auxiliary motion is described by

$$\ddot{\bar{r}} + \omega^2(t)\bar{r} = 0 \tag{158}$$

where the vector $\bar{r}$ has Cartesian components (x, y). Solving (158) radially and transversely using polar coordinates (ρ, θ), where $\rho = |\bar{r}|$, $x = \rho \cos \theta$, and $y = r \sin \theta$, we obtain

$$\ddot{\rho} - \rho\dot{\theta}^2 + \omega^2(t)\rho = 0 \tag{159a}$$

$$(1/\rho)(d(\rho^2\dot{\theta})/dt) = 0 \tag{159b}$$

The latter equation implies $\rho^2\dot{\theta} = l$, where l is the angular momentum constant. Now, using this result in (159a), one obtains the same equation as (15) with k replaced by l^2, and after using $x = \rho \cos \theta$, $p = \dot{x}$, the invariant I of (14) reduces to $I = (1/2)l^2$. This shows that the constancy of I is equivalent to the constancy of the angular momentum of the auxiliary motion.

(ii) Takayama[34] has discussed the physical meaning of the invariant (14) for a real system — the forced betatron oscillation seen in accelerator and storage rings. In this case the motion is described by an equation of the type (19), where $g(t)$ is the external TD force. The conserved quantity, obtained for this system in the form of an "affine" invariant, namely $\dot{x}_1x_2 - x_1\dot{x}_2 = \text{const}$, where x_1 and x_2 are arbitrary solutions of (13), does not serve any purpose. On the other hand, a physical interpretation is sought for a form similar to (14) using the concept of the equilibrium orbit well known in accelerator physics. In fact, the equilibrium orbit (u, v) is the particular solution of the canonical equations

$$(du/dt) = (\partial H/\partial v) = v \tag{160}$$

$$(dv/dt) = -(\partial H/\partial u) = -\omega^2(t)u + g(t) \tag{161}$$

where the Hamiltonian H has the form

$$H = (1/2)[p^2 + \omega^2(t)x^2] - g(t)x \tag{162}$$

The linear transformation $x = u + X$, $p = v + P$, converts the Hamiltonian (162) into the form (12) and the corresponding invariant into the form (14), for which the physical interpretation becomes easier.

(iii) Kaushal and Korsch[21] interpreted the invariant I as a mapping between two similar dynamical systems satisfying a definite force law. It is shown that the invariant I can be written in the form

$$I = (1/2)(\bar{k}(x/\bar{x})^2 + k(\bar{x}/x)^2 + (\bar{x}\dot{x} - \dot{\bar{x}}x)^2) \tag{163}$$

for the Hamiltonian

$$H = (1/2)[p^2 + \omega^2(t)\, x^2 + k/x^2] \tag{164}$$

with $\bar{x}$ as the solution of the auxiliary equation

$$\bar{x} + \omega^2(t)\bar{x} = \bar{k}/\bar{x}^3 \tag{165}$$

Note that (165) is actually the equation of motion corresponding to a similar Hamiltonian system

$$\bar{H} = (1/2)[\bar{p}^2 + \omega^2(t)\bar{x}^2 + \bar{k}/\bar{x}^2] \tag{166}$$

where k in (164) and $\bar{k}$ in (166) are constants. Thus, the equation of motion corresponding to $\bar{H}$ is the same as the auxiliary equation for H and vice versa. Here, I shows the correspondence between the Hamiltonians H and $\bar{H}$. In fact, the invariant (163) is a common constant of motion w.r.t. H and $\bar{H}$, so that we have

$$[I, H]_{\mathbf{PB}(x,p)} = [I, \bar{H}]_{\mathbf{PB}(\bar{x},\bar{p})} \tag{167}$$

i.e., I only generates a mapping between H and $\bar{H}$. It may be mentioned that in the spirit of the results of Section 4 [cf. (102)] the time dependence of k and $\bar{k}$ can also be investigated in such an interpretation of I, of course, for the corresponding 1D system.

(iv) An alternative interpretation of the invariant (163) can also be found by realizing that it has a Hamiltonian structure[66] in terms of the newly defined coordinate η and the time parameter τ. Note that I in (163) can be expressed in an alternative form as

$$I = (1/2)[k(x/\bar{x})^2 + \bar{k}(\bar{x}/x)^2 + \bar{x}^4((d/dt)(x/\bar{x}))^2] \tag{168}$$

Now, after defining $\eta = (x/\bar{x})$ and $\sqrt{2}\, d\tau = dt/\bar{x}^2$, $2I$ from (168) takes the form

$$2I = k\eta^2 + \bar{k}/\eta^2 + (1/2)(d\eta/d\tau)^2 \tag{169}$$

It is interesting to note that $2I$ here has a Hamiltonian structure with a potential term very similar to that of H or $\bar{H}$ [cf. (164) and (166)] and a kinetic term as $(1/2)(d\eta/d\tau)^2$.

8.2. Applications of Dynamical Invariants

In this subsection we highlight the applications of the knowledge of dynamical invariants (particularly that of the one derived for the TD HO system in 1D) in the context of quantum mechanics, the Feynman propagator (using the path-integral technique), cosmology, and relativistic TD Hamiltonian systems, obtaining the solution of a certain class of nonlinear differential equations, and of several other fields such as biophysics, plasma physics, and field theories.

8.2.1. Quantum Mechanics

The solution of the classical TD HO problem has suggested an alternative method for developing a Schrödinger-type quantum mechanics and a WKB-type semiclassical quantization condition. In particular, Korsch and his coworkers[62,63] and Lee[64] have used these methods to obtain the solution of some of the physical problems in an alternative manner. Some of these aspects are discussed here briefly.

8.2.1.1. Milne's Equation and WKB-Type Quantization. It can be seen that just a simple replacement of ρ, x, t, and $\omega(t)$ in the results for the TD HO system [cf. (12)–(15)] by w, u, x, and $k(x)$, respectively, leads to following equations:

$$w''(x) + k^2(x)w(x) = 1/w^3(x) \tag{170}$$

$$u''(x) + k^2(x)u(x) = 0 \tag{171}$$

If we use $k(x) = [2m(E - V(x))/\hbar^2]^{1/2}$, then (171) yields the form of the well-known Schrödinger equation, and (170) takes the form which is known as Milne's equation.[65] Now, if we know a particular solution of (170), the general solution of (171) can be written as[62]

$$u(x) = cw(x) \sin[\int^x w^{-2}(x')\,dx' - b] \tag{172}$$

where c and b are arbitrary constants. Alternatively, the general solution of (170) can be obtained in terms of linearly independent solutions u_1 and u_2 of (171) as[28,61]

$$w(x) = [Au_1^2(x) + Bu_2^2(x) + 2Cu_1(x)u_2(x)]^{1/2} \tag{173}$$

where A, B, C are constants related to the Wronskian W of u_1 and u_2 by $AB - C^2 = W^{-2}$.

Using the boundary condition on the wave function for the bound state to exist, one arrives at

$$\int_{-\infty}^{\infty} w^{-2}(x)dx = (n+ 1\ \pi) \qquad (n = 0, 1, 2, \ldots) \tag{174}$$

which is termed[62,64] *Milne's quantization condition* for the energy levels E_n. In the semiclassical limit ($\hbar \to O$) of Milne's equation [i.e., after neglecting the w'' terms in (170)] one obtains $w(x) \approx [k(x)]^{-1/2}$, a result valid in the classically allowed region but which breaks down at the classical turning points. In this case the condition (174) takes the form

$$\int_{-\infty}^{\infty} k(x)\, dx = \left(n + \frac{1}{2}\right)\pi \qquad (n = 0, 1, 2, \ldots) \tag{175}$$

where the missing $\pi/2$ term accounts for the contribution from the classically forbidden regions. Several applications of this new quantization rule and its possible generalization to the complex energy case are discussed by Korsch and Laurent[62] and Korsch *et al.*[63]

8.2.1.2. Quantum Mechanics as a Multidimensional Ermakov Theory. Lee[64] offers a new dimension to the application of the Milne quantization condition and the Ermakov theory mentioned above. In particular, the close connection between the classical Ermakov theory and Milne quantization condition is exploited to the extent of finding a common mathematical framework within which an explanation of both classical particle mechanics and quantum wave mechanics can be sought. As a result, a physical interpretation of the Ermakov invariant in terms of the theory of wave–particle duality is suggested by Lee.[64] While it may be desirable to pursue several other problems within this framework, the problems pertaining to quantum ray optics and the hydrogen atom have already been looked into by Lee. As a matter of fact in the multidimensional Ermakov theory, there seems to exist[66] a common basis for both the Schrödinger equation in quantum mechanics and the Riccati-type equation satisfied by the superpotential in supersymmetric quantum mechanics.

8.2.2. Feynman Propagator (Using the Path-Integral Technique)

The existence of an invariant for a dynamical system also simplifies the calculation of the Feynman propagator. In fact the Feynman propagator, while already containing the spirit of the quantum superposition principle via integration over various paths, provides an alternative route from the classical to the quantum description of a system. Lawande *et al.*[67–70] have studied in detail the role played by the invariants in the propagator theory, using a large class of potentials. It is noticed that a great simplification arises in this case if the invariant is assumed to be of second order in momenta. Further, explicit path-integral calculations have shown that the propagators in general admit expansions in terms of the eigenfunctions of the invariant-operator. This, in fact, allows the Feynman propagator to be expressed in terms of the eigenfunctions of the Ermakov invariant in an exact manner.

A variety of dynamical systems have been studied by Lawande and his coworkers[67] by extending the propagator theory to several dimensions of applications. For details we refer to the excellent review by Khandekar and Lawande.[68] Here, we just demonstrate the use of the Ermakov invariant in the calculation of the propagator for a simple system described by the Lagrangian

$$L = (1/2)\dot{x}^2 - (\ddot{\rho}\alpha/\rho - \ddot{\alpha})x + (\ddot{\rho}/2\rho)x^2 - (1/\rho^2)F((x-\alpha)/\rho) \quad (176)$$

and which possesses a second-order invariant

$$I(x, p, t) = (1/2)[\rho(p - \dot{\alpha}) - \dot{\rho}(x - \alpha)]^2 + F((x-\alpha)/\rho) \quad (177)$$

Here, $\rho(t)$, $\alpha(t)$, and $F((x-\alpha)/\rho)$ are arbitrary functions of their arguments. It is possible to write (176) in the form

$$L = (d\chi/dt) + L_0 \quad (178)$$

where L_0 is a new Lagrangian given by

$$L_0 = (1/2)\rho^2[(d/dt)((x-\alpha)/\rho)]^2 - (1/\rho^2)F((x-\alpha)/\rho)$$

and

$$\chi = (\dot{\rho}/2\rho)x^2 + ((\dot{\alpha}\rho - \alpha\dot{\rho})/\rho)x - (1/2)\int \rho^2[(d/dt)(\alpha/\rho)]^2\, dt$$

The Feynman propagator $K(x'', t''; x', t')$, defined as the quantum mechanical amplitude for finding a particle at the position x'' at the time t'' if the particle had been at x' at an earlier time t', is expressed by

$$K(x'', t''; x'\ t') = \int \exp((i/\hbar)\int_{t'}^{t''} L\, dt)\, \mathcal{D}x(t) \quad (179)$$

where $\mathcal{D}x(t)$ is the usual Feynman differential measure. After carrying out some lengthy calculations, the propagator K can be expressed as[69]

$$K(x'', t''; x', t') = (\rho'\ \rho'')^{-1/2} \exp[(i/\hbar)(\chi(t'') - \chi(t'))]\, \overline{K}_0(\xi'', \tau''; \xi', \tau') \quad (180)$$

where

$$\overline{K}_0(\xi'', \tau''; \xi', \tau') = \int \exp((i/\hbar)\int_{\tau'}^{\tau''} \overline{L}_0\, d\tau)\, \mathcal{D}\xi(t)$$

with

$$\bar{L}_0 \equiv \bar{L}_0\,(\xi, d\xi/d\tau) = (1/2)(d\xi/d\tau)^2 - F(\xi) \qquad (\xi = x/\rho)$$

Thus, it becomes clear that the propagator for the TD system is related to the propagator for an associated TID system corresponding to the Lagrangian $\bar{L}_0$ in the new space-time variables (ξ, τ). Further, note that the invariant (177) is basically the Hamiltonian $\bar{H}_0$ associated with the Lagrangian $\bar{L}_0$. Khandekar and Lawande[70] have also obtained the Feynman propagator in an exact and closed form for NC potentials.

8.2.3. *Cosmological Applications*

The importance of dynamical invariants in the fields of cosmology and astrophysics is well known.[11,26] In particular, Berger,[71] Misner,[72] and Ray[73] have studied in detail the relationship between the particle number present in a cosmological model and an adiabatic invariant. The knowledge of invariants, in fact, offers an alternative for calculating the particle production in cosmological models.

Here we discuss rather briefly the related adiabatic invariant, which is defined as a semiclassical particle number, and the alternative method for calculating the particle production in models with an initial singularity. Once the particle number for each mode of the field is defined as an adiabatic invariant, then it becomes interesting to relate this quantity to the parameters of the field mode near the initial singularity. Following Berger,[71] the amplitude $\phi_{\mathbf{k}}$ for each mode $\mathbf{k}$ of a minimally coupled scalar field of mass m satisfies

$$\ddot{\phi}_{\mathbf{k}} + \omega_k^2\,(\tau)\phi_{\mathbf{k}} = 0 \tag{181}$$

where τ is a new time coordinate. In a regime in which the expansion or contraction time scale of the universe greatly exceeds the period of the mode $\mathbf{k}$, the solution to (181) can be obtained[71] using the WKB approximation as

$$\phi_{\mathrm{WKB}} \approx \omega^{1/2} A\cos\left(\int^{\tau} \omega\, d\tau' + \zeta\right) + \omega^{-1/2} B\sin\left(\int^{\tau} \omega\, d\tau' + \zeta\right) \tag{182}$$

where ζ is a constant phase and A, B are arbitrary constants. If the energy of the mode $\mathbf{k}$ is formally defined as

$$E\,(t) = (1/2)\,(\dot{\phi}^2 + \omega^2(t)\phi^2) \tag{183}$$

then in the WKB limit there exists[13] an adiabatic invariant N of the type $N = E_{\mathrm{WKB}}/\omega$, which in turn reduces to the form

$$N = (1/2)(A^2 + B^2) \tag{184}$$

On the other hand, Ray[73] makes use of the available form of the invariant for the system (183) as

$$I = (1/2)[(\phi/\rho)^2 + (\rho\dot{\phi} - \phi\dot{\rho})^2] \tag{185}$$

with ρ satisfying

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3} \tag{186}$$

He, as before, identifies N of (184) with I in the adiabatic regime. In a model that possesses an initial singularity at time $t = t_s$ such that $\omega(t_s) = 0$, the invariant is expected to take the same form as (185) with ϕ and ρ replaced by ϕ_s and ρ_s corresponding to $t = t_s$, and the solutions of (181) and (186) for this case are expressed as[73]

$$\phi = q_0 + p_0\,(t - t_s); \qquad \rho = (1/\alpha)[1 + \alpha^4\,(t - t_s + (\beta/\alpha))^2]^{1/2} \tag{187}$$

where q_0, p_0, α and β are integration constants. Further, use of these results yields the invariant (185) in the form

$$N = (1/2)[\alpha^2\, q_0^2 + (1/\alpha^2 + \beta^2)p_0^2 - 2\alpha\beta q_0 p_0] \tag{188}$$

Clearly, the modified definition of the constants A and B in (182) as $A = p_0/\alpha$ and $B = \alpha q_0 - \beta p_0$ shows the equivalence of the forms (188) and (184). Thus, the Ermakov invariant provides an interesting alternative for calculating the particle production in cosmological models. Since it is an exact invariant, its leading term in the adiabatic series is the particle number.

8.2.4. Relativistic TD Hamiltonian Systems

The role of classical dynamical invariants is also found to be important in some special types of relativistic TD systems. Recently[74] there has been an attempt to obtain the invariant for the system described by the Hamiltonian

$$H = (p^2 + 1)^{1/2} + V(x, t) \tag{189}$$

in the form $I(x, p, t) \equiv \Psi(V(x, t), p)$. The search for integrable systems admitting such an invariant, following the method of Giacomini,[75] reduces to the solution of the "potential" equation of the form

$$(dV/dF_1) + f(V) + F_1/(F_1^2 + 1)^{1/2} = 0 \tag{190}$$

where both F_1 and f are arbitrary functions of V and are related. The invariants corresponding to the three cases, namely $f(V) = 1$, $f(V) = V$, and $f(V) = \exp(V)$, are obtained by Martin and Bouquet,[74] respectively, as

$$I = P + \mathscr{V} + (p^2 + 1)^{1/2} \tag{191}$$

$$I = V(x, t) \exp(p) + \int_0^p du \cdot u \exp(u)/g(u)$$

and

$$I = \{\exp[-V(x, t)] - p\} \exp[-(p^2 + 1)^{1/2}] - \int_0^p du \cdot u^2 \exp[-g(u)]/g(u)$$

where $V(x, t) = \mathscr{V}(x + t)$ and $g(u) = (u^2 + 1)^{1/2}$. As an example, the case of the TD relativistic HO system in the form

$$H_{\mathrm{R}} = (p^2 + 1)^{1/2} + x^2/2t^2 \tag{192}$$

has been analyzed. Interestingly, such a system does not turn out to be analytically integrable, mainly due to the special nature of the kinetic term in H_{R}.

8.2.5. Solution of a Class of Nonlinear Differential Equations

Knowledge of invariants also helps in investigating the solution of a particular class of nonlinear differential equation. Note that for the system (damped TD HO)

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0 \tag{193}$$

the invariant turns out to be

$$I = h^2(x/\rho)^2 + (\dot{\rho}x - \rho\dot{x}) \exp\left[2 \int_0^t P(t)\, dt\right] \tag{194}$$

with $\rho(t)$ satisfying

$$\ddot{\rho} + P(t)\dot{\rho} + Q(t)\rho = (h^2/\rho^3) \exp\left[-2 \int_0^t P(t)\, dt\right] \tag{195}$$

Eliezer and Grey[61] put these simple results in the form of the following theorem:

Theorem. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0 \tag{193$'$}$$

then the general solution of

$$y'' + P(x)y' + Q(x)y = (h^2/y^a)\exp\left[-2\int P(x)\,dx\right] \tag{195'}$$

may be written as [cf. (173)]

$$y = (Ay_1^2 + By_2^2 + 2Cy_1y_2)^{1/2} \tag{196}$$

Here, A, B, and C are arbitrary constants and are related through $AB - C^2 = (h^2/W^2)\exp[\int P(x)\,dx]$, with $W = y_1y_2' - y_1'y_2$ defined as the Wronskian for (193′). Clearly, in (193) and (195′) the dependent and independent variables are redefined and accordingly the dots are replaced by primes in the definition of the derivatives.

Pinney[54] studied a nonlinear equation of the type

$$y'' + P(x)y' + Q(x)y = ABW^2y^{-3} \tag{197}$$

which again is the modified version of the auxiliary equation (15) corresponding to a damped TD HO. The solution to (197) is found to be $y = (Ay_1^2 + By_2^2)^{1/2}$. Several generalized versions of the solution (196) to (195′) have been investigated.[54,76–78] For example, Reid[76] obtained the solution of the differential equation

$$y'' + P(x)y' + Q(x)y = AB(m-1)(y_1y_2)^{m-2}W^2y^{1-2m} \tag{198}$$

in the form $y = (Ay_1^m + By_2^m)^{1/2}$, where m is a constant assumed to be real and nonzero. Thomas[77] has shown that the function $y = (y_1y_2)^{k/2}$ satisfies the nonlinear equation

$$y'' + P(x)y' + kQ(x)y = (1-l)y'^2y^{-1} - (1/4)kW^2\,y^{1-4l} \tag{199}$$

where k is assumed to be real and nonzero and $kl = 1$. As a next step of generalization of (195′), the function

$$y = (Ay_1^m + By_2^m + mCy_1^jy_2^n)^{k/m}$$

where $m = j + n$, is found to represent the solution of a nonlinear differential equation of more general type, namely

$$y'' + P(x)y' + kQ(x)y = (1-l)y'^2y^{-1} + kUW^2y^{1-2ml} \tag{200}$$

where $ki = 1$ as before, and

$$\begin{aligned} U = {} & Cy_1^{j-2}\,y_2^{n-2}\,[(m-j-1)\,nAy_1^m + (m-n-1)jBy_2^m - Cnjy_1^jy_2^n] \\ & + (m-1)AB(y_1y_2)^{m-2} \end{aligned}$$

It is obvious that the earlier equations can be obtained as special cases of this last equation. For further details of such studies we refer to the literature.[54,76–78] Interestingly, these classical dynamical studies have also suggested some clues for the solution of several quantum problems. In particular, Burt and Reid[78] have studied the solution of a nonlinear Klein–Gordon equation.

So far in this subsection we have discussed the solution of nonlinear equations of the type (200) in terms of the solutions of (193′). But it may be recalled here that we have already investigated in detail in Section 6 another dimension of applications of the knowledge of invariants in relation to coupled nonlinear differential equations in terms of Ermakov systems.

8.2.6. Other Applications

In this subsection we briefly highlight the role which dynamical invariants can possibly play (and, in fact, have already been playing in some localized domains) in the fields of biophysics, plasma physics, and field theories. The present survey of TD systems can help further a better understanding of the respective phenomenon in these areas.

8.2.6.1. Biophysics. In neurophysiology different models have been proposed for nerve impulse propagation in terms of nonlinear PDEs. The common difficulty with these models is that they turn out to be nonintegrable,[79,80] In the following we briefly discuss a generalized but completely integrable model for nerve impulse propagation proposed and pursued by Rajagopal[81,82] in a series of papers.

The basic equation governing nerve impulse propagation is of the form of a nonlinear diffusion equation which, after introducing a phenomenological expression for the ion current, can be written as

$$V_{xx} - V_t = F(V) \tag{201}$$

where $V(x, t)$ is the voltage across the membrane and V_t is the measure of the displacement current per unit length passing through the membrane. Here and in the following the subscripts indicate the variables w.r.t. which the partial derivatives are taken. Somehow (201) fails to reproduce the important feature of pulse recovery which is necessary for a repeated firing of the fiber. An account of such a recovery variable R yields (201) in the form

$$V_{xx} - V_t = F(V) + R_t \tag{202}$$

with $R_t = \varepsilon(V + a - bR)$. Here ε is proportional to the temperature factor and a and b are constants.

In his generalization of the model Rajagopal uses two first-order differential equations in place of (202). For a space-clamped axon, it is assumed that the rate of change of membrane potential (X_t) depends linearly on Z (the current stimulus applied through the electrode to the axon) and on Y (the intrinsic current) and nonlinearly on the membrane potential X, thereby implying

$$X_t = -a' \, [F(X) - Y - Z] \tag{203}$$

Further, the rate of change of intrinsic current (Y_t) is chosen as $Y_t = b'[G(X) - Y]$. Here a' and b' are arbitrary constants. Although the experiments[79,80] suggest a cubic nonlinearity for $F(X)$ and an exponential one for $G(X)$, Rajagopal[82] assumes the same form for $F(X)$, $Z(t)$, and $G(X)$ as

$$F(X) = k_1X - k_2X^3; \quad Z(t) = k_3X - k_4X^3; \quad G(X) = \alpha X - \beta X^3 \tag{204}$$

where k_i ($i = 1, 2, 3, 4$) are arbitrary constants and $\alpha = k_1 - k_3$, $\beta = k_2 - k_4$. After a straightforward derivation, the ODE to be handled for the case of traveling waves turns out to be

$$d^2X/d\xi^2 = \gamma X - \delta X^3 \tag{205}$$

where $\gamma = \alpha a' + b'$ and $\delta = \beta a'$, and $\xi = x - ut$ is the moving space variable with u as the wave velocity. Although the solution of (205) has been obtained explicitly in terms of elliptic functions, an account of the time dependence of γ and δ (depending upon the choice of the model) in (205) will bring in the knowledge of dynamical invariants discussed in Section 2.2.

8.2.6.2. Plasma Physics. Whenever the time-varying (electromagnetic or simply magnetic) field is involved in understanding (theoretically as well as experimentally) a physical phenomenon, the corresponding dynamical system turns out to be a TD one. Further, the invariants may exist and can be constructed for such a system in an exact or an approximate manner depending upon the nature of the time dependence. Such systems have been known[12] for a long time in the fields of plasma physics and so-called magnetic surfaces. We make the such discussion brief here. Only recently, the representation of a magnetic field with toroidal topology in terms of field-line invariants has been studied by Lewis[83] and Lewis and Abraham-Shrauner.[84] They use Boozer's representation[85] for a magnetic field with toroidal topology in the form

$$\mathbf{B}(\mathbf{r}) = (\nabla\psi_0) \times (\nabla\theta_0) + (\nabla\phi_0) \times (\nabla\chi_0) \tag{206}$$

where $\theta_0(\mathbf{r})$, $\phi_0(\mathbf{r})$, $\chi_0(\mathbf{r})$, and $\psi_0(\mathbf{r})$, respectively, are the poloidal angle, toroidal angle, poloidal flux, and toroidal flux functions which characterize the nature of the magnetic field. Note that any $\mathbf{B}(\mathbf{r})$ given by (206) is divergence-free, i.e., $\nabla \cdot \mathbf{B} = 0$. The main advantage of using (206) is that it allows direct representations in Hamiltonian form of the differential equation for the magnetic field lines. Thus, the representation (206) turns out to be useful for applications to tokamaks and stellarators. For the details we refer to refs. 83 and 84.

8.2.6.3. Field Theories. The classical analogue of quantum field theories in a suitable choice of the gauge also gives rise[86] to dynamical systems. Savvidy[86] studied the Yang–Mills system described by the Lagrangian in the covariant form as

$$\mathcal{L} = -(1/4)\, F^a_{\mu\upsilon}\, F^a_{\mu\upsilon}$$

in the $SU(2)$ case by resorting to the gauge $A^a_0 = 0$, $A^a_i = A^a_i\,(t)$. Here the symbols have their usual meanings. This has led to a classical dynamical system described by the coupled nonlinear differential equations of the form

$$\ddot{f}^a + (1/2) \sum_b (f^b)^2 f^a = 0$$

Further simplifications are achieved[87] by imposing an additional condition $A^0_3 = 0$.

Recently, the Abelian Higgs model described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\upsilon} F_{\mu\upsilon} + \tfrac{1}{2}(D_\mu \phi)^* \,(D^\mu \phi) + C_2 |\phi|^2 - C_4 |\phi|^4$$

has been studied by Kumar and Khare[88] in (2 + 1) dimensions. They make an ansatz for the gauge and Higgs fields as

$$A_0(\mathbf{x}, t) = 0, \qquad A_1(\mathbf{x}, t) = A_2(\mathbf{x}, t) = h\,(t)/\sqrt{2}$$
$$\phi(\mathbf{x}, t) = \exp(i\omega(x + y))\; q_2(t)$$

Further, by defining $h\,(t) = q_1(t) + \sqrt{2}\,\omega/e$, these authors obtain the system of nonlinear equations

$$\ddot{q}_1(t) = -e^2 q_1 q_2^2; \qquad \ddot{q}_2(t) = 2C_2 q_2 - 4C_4 q_2^3 - e^2 q_1^2\, q_2 \tag{207}$$

Clearly, such a system of equations can offer the example of an Ermakov-type system under certain conditions.

Besides the above areas where the scope of invariants has been highlighted, the role of dynamical invariants in the fields of condensed matter physics, statistical mechanics, and astrophysics has been well known in the literature. We again refrain from these discussions here.

9. CONCLUDING DISCUSSION AND FUTURE PROSPECTS

For the purposes of constructing exact invariants for systems involving an explicit time dependence, a brief survey of various methods used in the literature has been carried out. The number of methods used for 1D systems is still more than those used for 2D systems. Though it is easier to test the merit of a new method on a 1D system, not all the methods used for this purpose have been applied to two or higher dimensional systems. In fact, the underlying intricacies of a method reflect much more when it is applied to higher dimensional systems. Sometimes the method also shows limitations when it is extended to higher dimensions.

As such, TD systems in 3D have not been studied to the same extent as TID ones. Even for the TID case only a few attempts[59,89] have been made, and in some cases[59] the assumption of a certain type of symmetry in the system reduces the problem to the 2D case. As a matter of fact there appears a lot of simplification in the construction of invariants for TID systems since the ansatz for I [cf. (56)] in this case is allowed[2,3] to contain only either even powers or odd powers in momenta as a result of the time-reversal symmetry. However, this is not possible with TD systems and this makes their study more difficult. It may appear trivial to extend almost all the methods used for 1D and 2D systems (cf. Sections 3 and 4) to the 3D case, but, as mentioned before, the complexity in the construction of invariants increases not only with the order of invariants, but also with the dimensions. To visualize the degree of complexity in the 3D case, for example, we mention the following: in the rationalization method, while the summations on the i_i in the ansatz (56) run from 1 to 3 and (58)–(61) remain intact in the 3D case, the number of these latter equations to be handled finally in component form turns out to be 20 for the second-order invariants, whereas in the 2D case this number is just 10. Similarly, for the third-order invariants in the 3D case, the number of equations to be handled turns out to be 36 compared to 15 for the 2D case (cf. Section 5.2). Indeed such complexity is a common feature and appears[93] in one form or another in all the methods when they are extended to 3D systems. Here we briefly highlight the 3D problem of coupled oscillators in its general form, as an extension of the 2D results (cf. Section 4.2), within the framework of the dynamical algebraic approach, and point out some underlying intricacies in this particular case.

We consider the Hamiltonian[93]

$$\begin{aligned} H = {}& (1/2)(p_1^2 + p_2^2 + p_3^2) + \alpha_1(t)x_1^2 + \alpha_2(t)x_2^2 \\ & + \alpha_3(t)x_3^2 + \beta(t)\phi(x_1, x_2, x_3) \end{aligned} \tag{208}$$

In order to express (208) in the form (86), we identify the Γ_n and h_n,

$$\Gamma_1 = (p_1^2/2); \qquad \Gamma_2 = (p_2^2/2); \qquad \Gamma_3 = (p_3^2/2)$$

$$\Gamma_4 = x_1^2; \qquad \Gamma_5 = x_2^2; \qquad \Gamma_6 = x_3^2; \qquad \Gamma_7 = \phi(x_1, x_2, x_3)$$

$$h_1 = h_2 = h_3 = 1; \qquad h_4 = \alpha_1(t)$$

$$h_5 = \alpha_2(t), \qquad h_6 = \alpha_3(t), \qquad h_7 = \beta(t)$$

and compute the nonvanishing Poisson brackets $[\Gamma_m, \Gamma_n]_{PB}$. While the number of nonvanishing Poisson brackets now turns out to be large compared to the 2D case, one needs to introduce three more Γ_n to close the Lie algebra, namely $\Gamma_8 = -2p_1x_1$, $\Gamma_9 = -2p_2x_2$, and $\Gamma_{10} = -2p_3x_3$ with the corresponding h_n as $h_8 = h_9 = h_{10} = 0$. Finally, for a system admitting a second-order invariant, the equation satisfied by ϕ in the present 3D case becomes

$$\delta(t)\phi + x_1(\partial\phi/\partial x_1) + x_2(\partial\phi/\partial x_2) + x_2(\partial\phi/\partial x_3) = 0 \tag{209}$$

with $\delta(t)$ as defined before by (101). Now, it is not difficult to obtain two special solutions of (209) in the form [cf. (100)]

$$\phi(x_1, x_2, x_3) = k_1 x_1^{-\delta} + k_2 x_2^{-\delta} + k_3 x_3^{-\delta} \tag{210a}$$

$$\phi(x_1, x_2, x_3) = k_4(x_2/x_1)^{c_0}\, x_1^{-\delta}(x_3/x_1)^{c_0'} + k_5(x_3/x_2)^{c_0}\, x_2^{-\delta}\,(x_1/x_2)^{c_0'}$$
$$+ k_6\,(x_1/x_3)^{c_0}\, x_3^{-\delta}\,(x_2/x_3)^{c_0'} \tag{210b}$$

where c_0 and c_0' are the separation and k_i ($i = 1, \ldots, 6$) are the integration constants. The first invariant for these two forms of ϕ in (208) can be written down easily in accordance with the results of Section 4.2. The important point to be noted here is about the nature of the rationale regarding the powers of the coupling term in (208). This is also retained here in the same form as in the 2D case. In other words, if we express a particular term in ϕ in (210b) as $\phi = kx_1^m x_2^n x_3^l$, then the first invariant is found to exist only for the case when $m + n + l = -\delta$, where δ is a constant and equals 2, as before in the 2D case. Further, for the existence of this invariant there occurs one more constraining relation, namely $(\dddot{\psi}/4) + 2\alpha_3\dot{\psi} + \dot{\alpha}_3\psi = 0$, in addition to (97a) and (97b). As a matter of fact, the system (210b) will lead to Ermakov-type systems in higher (3 + 1) dimensions.

In spite of so many methods for 1D systems, not many new systems have been found for which the invariants can be constructed. In this respect very often the TDHO system with its possible generalizations has played a pivotal role. In fact while the TDHO has offered a testing ground for various methods, the TD anharmonic oscillator problem even in 1D could not be investigated in most of the methods except for the one used by Leach and his coworkers.[24,25,27] Some of the approaches (like the dynamical algebraic one or the transformation-group method, for that matter), have an obvious capability from the point of view of extending them to the corresponding quantum domain, but this would be without actually demonstrating their complete potential at the classical level. Although the inherent mathematical elegance in the dynamical algebraic approach has beautifully suggested the criterion for the relative time dependence of various coupling terms in the 2D case (cf. Section 4.2), it somehow does not help in providing the second invariant for these systems if it exists. There remains, however, the difficulty of closure of the algebra as one proceeds not only for obtaining the higher order invariants in this approach, but also for accounting for the higher order anharmonic terms with positive powers in the $V(x_1, x_2, t)$. For this purpose, it may be of interest to use the available[90] generalized versions of this method.

Whether it is the 1D or the 2D case, we have not looked into very general solutions of the derived potential equations [cf. (29), (65), (78)–(80), and (106)] in rigorous mathematical terms except for trying some of their particular solutions, which are often assumed to be separable in coordinates and time variables. Although it is difficult to obtain such general solutions, if one obtains them, then they may provide several new systems which have not been covered in this paper. The potential equation (81), derived using complex coordinates for the 2D case, has a special status in the sense that it offers the invariants for TD central potentials of nonharmonic nature.

In view of the fact that one of the coefficient functions and subsequently the potential function $V(x, t)$ in the rationalization method in the 1D case factorizes[29] in x and t variables in a rather natural manner [cf. (107) for the coefficient function $b_1(x, t)$, with a similar situation for $b_0(x, t)$ in the second-order case], the role of the self-similar transformation (49) in giving rise to the form (52) remains questionable. Anyway, the use of the self-similar technique, while it suggests an easier method to derive the higher order invariants for the 1D case, it does not provide very different integrable systems as far as their functional forms are concerned.

Mention may be made of other functional forms of the invariants used for TD systems. In this review, while our survey mainly concerns the polynomial (in momenta) forms, nonpolynomial forms for the TD systems have not been as frequently studied as TID systems. There has also been discussion of the complex[91] invariants for the TDHO system. However, such ideas are more relevant in the context of quantum mechanics. In fact, in this case one exploits the generalized canonical transformations of Lewis and Leach[92] to convert the TD system into a corresponding TID one and then looks for the invariants in the Heisenberg picture.

Goedert and Lewis[7] have recently studied the rational form of the invariant for 1D TD systems, using the momentum-resonance formulation of Lewis and Leach.[7] For the system (26), in this case one makes an ansatz for the invariant as

$$I(x, p, t) = c(x, t) + \sum_{n=1}^{N} \frac{v_n(x, t)}{p - u_n(x, t)}$$

Here, I is a rational function of momentum with simple poles, which are called momentum-resonances. As before, the use of (5) and (6) leads to necessary and sufficient conditions on the functions c, v_n, and u_n and subsequently one expresses the resonance type invariant I as a functional of the potential $V(x, t)$. The cases of one, two, and three resonances corresponding, respectively, to the values 1, 2, and 3 of N are investigated. Although we have tried to obtain the conditions in general under which the second invariant for 2D TD systems can be constructed in a polynomial form (cf. Section 7), the available methods are inadequate for this purpose. Perhaps the second invariant for such systems could be of nonpolynomial form, which, of course, has not been studied so far.

Finally, the important observation made already in connection with studies of coupled oscillators, in both TD (cf. Sections 4.1 and 4.2) and TID cases,[41] concerns the powers of the coupling terms in the Lagrangian, i.e., the terms of the type $x_1^m x_2^n$ [cf. (70) and (146)]. Interestingly, it is found that this system admits an invariant only for the case when $m + n = -2$. Some of the integrable systems corresponding to this situation are listed in Table 1 of ref. 41), and the time dependence of the others is discussed here in Sections 4.1, 4.2, and 6.1 mainly in the context of generalized Ermakov systems. Although such a restriction on m and n appears in a natural manner in the rationalization method and is also found to have a basis in the closure of the dynamical algebra of phase space functions, the question remains as to why there is only this peculiar restriction. On the other hand, the ad hoc choices of m_1 and m_2 made by Ray and Reid[28] in their work [cf. (138)] also conform to such a restriction on m and n. From all this it appears that this restriction indeed has a much deeper origin than merely providing the integrable systems. It could well be that the special setting of the kinetic and harmonic terms in the potential in the Hamiltonian structure is responsible for such a restriction.

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